Ways to Think About Mathematics:
Activities and Investigations for Grade 6-12 Teachers

Overview of the materials

Professional development for teachers has always been important in keeping them fresh, up to date, and intellectually alive. Increasingly, the calls for professional development have been backed by state requirements and by programs funded, organized, or run at national, state, and local levels. *Ways to Think About Mathematics: Activities and Investigations for Grade 6-12 Teachers* was written with a very particular perspective on the content and structure of professional development in secondary mathematics, grades 6 to 12. Our experience—as teachers, teacher educators, and mathematicians ourselves—has been that the most effective professional development in mathematics for teachers is immersion in the mathematics itself, and the most engaging and ultimately useful mathematics is the very mathematics that the teachers use or teach every day. There is depth in that mathematics, even at the sixth-grade level—one need not go beyond it to find new ideas and fresh connections. There are also important connections between that mathematics and the college mathematics courses that teachers may have taken and that often seem so very far removed from what they do in class on a day-to-day basis.

With these two kinds of connections in mind—connections within the mathematics that teachers teach, and connections between that mathematics and the seemingly more abstruse courses they may have taken—we proposed the *Connecting with Mathematics* project to the National Science Foundation. The materials created by the *Connecting with Mathematics* project team led to this book.

Each of the chapters takes problems directly from secondary curricula, lets teachers plumb the mathematics behind them through their own problem-solving and investigation, and emerges in some unexpected mathematical territories along the way. The course remains practical because it is focused on the mathematics of the classroom, and though the problems are designed for teachers, many of them can be adapted for classroom use.

One chapter explores the notion of mathematical investigation itself—the process of modifying a problem in various ways to explore its edges and depth without changing it so radically that one is simply off topic, and the process of moving from an experimenting, idea-gathering, and conjecturing stage to a stage in which one systematizes and explains through proof. Other chapters explore particular topics, including ones in algebra, combinatorics, geometry and measure, and number theory.

Outlines of the chapters

- **What is Mathematical Investigation?**
  In this chapter, participants take a mathematical investigation (which counting numbers can be expressed as the sum of at least two consecutive counting numbers?) from start to finish—from exploratory stages through reporting logically connected results—learning strategies that they can use with their students to develop their investigative skills. New mathematical facts and relationships may well be encountered, but the real purpose of this chapter is to investigate investigation.
- **Dissections and Area**
  This chapter provides a new approach to learning and teaching area, and mathematical ideas related to area. Participant teachers will rethink the concept of area through hands-on activities that involve the dissection and rearrangement of various geometrical figures. Using dissections, they will derive area formulas for various polygons and explore the meaning and properties of area in depth. This chapter also uncovers connections between area, geometrical transformations, algorithms, functions, coordinate geometry, and fractals. Interesting applications, such as tessellations of the plane, tangrams, and jigsaw puzzles, are discussed.
- **Linearity and Proportional Reasoning**
  This chapter explores the gamut of proportional thinking from its elementary origins through generalizations in several directions and dimensions. The first section explores absolute and relative comparisons. The second section links sequences with interpolation. The third section explores functions satisfying the property $f(a+b) = f(a) + f(b)$ (for all a and b). Participants will ask, for which functions is this actually true? In the fourth section, functions of two variables emerge from old-fashioned and new style word problems, and their graphs are investigated. The final section explores higher dimensional analogues of linear functions.

Did you know that the book The Pythagorean Proposition *contains* 366 *proofs of the Pythagorean Theorem?*

- **Pythagoras and Cousins**
  This chapter delves into the Pythagorean Theorem and related topics. Some of the topics include various proofs and generalizations of the Theorem and properties of Pythagorean triangles (right triangle with integer-valued legs and hypotenuse). Some proofs involve dissecting and reconstituting areas (as in the Dissections chapter), while others are chosen for their originality, beauty, or historical context, or to illustrate a method or idea that will be useful later. Another section asks teachers to derive an algorithm for producing all Pythagorean triples [ordered triples of positive integers (a, b, c) so that $a^2 + b^2 = c^2$] and use this algorithm to solve a variety of problems. The last section brings teachers back to the classroom to investigate the mathematics behind creating problems that "come out nice."
- **Pascal's Revenge: Combinatorial Algebra**
  Participants engage in combinatorial investigations while they work on concrete problems that encourage them to search for patterns and to create and confirm/disprove conjectures about more general situations. Through hands-on activities, teachers reflect on and discuss a variety of mathematical contexts all tied together by Pascal's Triangle and binomial coefficients. In the first section, participants work with partitions of a positive integer n (the number of different ways to express n as a sum of positive integers) through activities with number rods. Section 2 uncovers facts about graphs on the coordinate plane (specifically taxicab geometry) by asking questions like, How many ways are there to go a total of 5 blocks north and 8 blocks west? In sections 3 and 4, participants will discover explicit connections between Pascal's Triangle, binomial coefficients, and the Binomial Theorem, while section 5 introduces the capstone notion of generating functions, introduced through an analysis of dice rolls.

Organization of this book

The materials come in three pieces:

- *Activities and investigations:* Each chapter of the main text contains 5 sections, each consisting of 5-10 pages of problems for teachers to work on. Many sections start with a rich exploratory problem from one of the standards-based curricula. They continue with related problems that lead teachers to explore new mathematical territories or delve more deeply.

- *Problems for the classroom:* It is generally impossible for teachers to take off their "teaching hat," but it is important to keep in mind that, even though many of the problems might be appropriate—or adaptable—for student work, the text problems are designed for *you* to ponder, struggle with, talk about, and eventually solve with the help of your fellow participants and facilitator. As a reward for your participation, the *Problems for the Classroom* section includes problems, with solutions, for use in grades 6-12 classrooms.
- *Answers to selected problems:* Where appropriate, brief answers are given to some of the text activities. Complete solutions to all of the problems are available on the supplementary CD-ROM (see page xi for more information).

How to use this book

The materials in this book, the product of the *Connecting with Mathematics* project at Education Development Center, are designed to give you an experience with exploratory mathematics. You will work on a number of problems designed to get you thinking, conjecturing, and problem solving (i.e., *doing* mathematics). Being an effective teacher is one of the hardest jobs there is, and mathematics is a topic that has long been seen as difficult to teach and learn. A good mathematics teacher is part child psychologist, part motivational speaker, part mathematician, and any number of other "parts"! While the goal is to help you become an even better *teacher*, the strategy of the materials is to get you to do mathematics. In experiencing open-ended problems, making and checking conjectures, and evaluating your own problem-solving strategies, you become better prepared to deal with day-to-day classroom decisions:

Creation of the Connecting with Mathematics *materials was made possible by grant #ESI-9731244 from the National Science Foundation.*

> *Hmm, some of my students are approaching this problem in a different way than I expected. Is it OK to let them pursue this line of reasoning? Thinking through some of the possibilities, I see that this might be a fruitful pursuit. Should I pursue the line of thinking that Sally suggests, or will that lead us toward a dead end? Maybe that dead end will serve as a "jumping off" point for another activity. Or will it just leave them running around in circles and cause too much frustration?*

Sometimes, a little frustration is a good *thing. It can motivate students (and ourselves) to dig deeper to gain more understanding.*

Some directions are fruitful to pursue; some are dead ends not worth spending time on (although sometimes it *can* be valuable for students to see a dead end, just to know that there are strategies for backing up and taking another track). To make sound pedagogical decisions for your class, you need to be sensitive to the needs of your class, but you *also* need to know how to recognize the path markers along the problem-solving trail. This involves familiarity with doing mathematics. Confidence in your own mathematical understanding of the topic, gained through *doing* mathematics, will help you make these pedagogical decisions based on what is best for your class that day.

The sections within each chapter are variations on a single theme (which varies from chapter to chapter), and we encourage you to think about the connections between problems, sections, and chapters. Some of these connections are obvious, while others are quite subtle. In any case, it is essential to keep in mind that the problems have been carefully designed to tell a *story*. It is therefore important to work on all of them—in the order given—even if you think you know the answers already.

Some problems might require extra reflection and/or discussion time, in which case they are formatted as follows:

Reflect and Discuss

2. With your partner(s), devise a set of directions that explains how to *list* all trains of any given length.

Other "big picture" problems are further set apart (with a title and special formatting) as illustrated below:

PROBLEM

3. THE TRAIN PROBLEM:
Given any positive integer, n, derive a formula for the number of trains of length n. Explain your solution, the process you used to find it, and how you know it's correct *for all* n.

In other situations, we provide a problem — or problems — that provide practice with a newly discussed concept:

Check Your Understanding

Pas(n,k) *is used to denote entry* k *of row* n *in Pascal's Triangle, which is defined recursively.*

2. Compute $Pas(6,2)$ and $Pas(8,5)$ using only the definition.

3. Use the $Pas(n,k)$ notation to *express* the fact that each row of Pascal's Triangle is a *palindrome* (that is, it reads the same forward and backward). More to the point, what other entry in the n^{th} row of Pascal's Triangle is equal to $Pas(n,k)$?

4. Express the hockey stick (or sock) property (mentioned in problem 5 in section 2) in "Pas" notation.

Ways to think about it

If you stumble or hit a roadblock, you can refer to *Ways to think about it* for ideas to help you get back on track. *Ways to think about it*, included at the end of each section, provides suggestions and additional questions to help you organize strategies for solving the problems posed during the class or session. Be sure to read it when you are finished. We also hope that you'll find the marginal notes useful. They usually provide supplementary information (or an occasional humorous anecdote), but when we *really* want you to read it, we will provide one of the visual clues ($\gtrdot$ or $\lessdot$) to remind you to "look over" at the note in the margin. So take a few minutes to get to know the members of your group and then dive in—the water's fine!

If you'd rather dip your toes in and slowly edge your way in, that's OK, but you'll get wet sooner or later!

Supplementary materials

The *Facilitator's Guide* offers a variety of support materials to help facilitators implement the text. Each section of the *Facilitator's Guide* typically includes

- An overview section that describes the main focus and the story line of a section and puts them into the context of the whole chapter.
- Specific guidance on problems presented in the text that describes the goals and the purposes of the problems, suggests possible approaches to the problems, and gives a facilitation timeframe.

The *Further Exploration CD-ROM* (accompanying the Facilitator's Guide) extends the content of the main text with materials that can serve as a resource for independent work between classes, or as a reading outside of professional development programs. Besides additional readings for teachers, these materials also feature:

- Additional activities and explorations for teachers
- Solutions to all problems from the text and the *Further Exploration* materials.

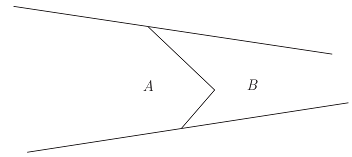
Index of key problems

Some problem statements have been slightly modified in order to provide context.

It will usually be useful to also consider the activities before and after these problems, since the surrounding explorations include essential concepts and strategies for solving—or applications of—the key problems.

Since this book is a structured collection of mathematical explorations, it is less encyclopedic than the usual textbook. Though we believe it is important to work through chapters and sections in the order the problems are presented (so that the "story line" is not lost), we realize that there might be compelling reasons to review on one or more of the big picture problems. Therefore, in lieu of an index, we provide here an annotated list of the key problems presented in the text (those labeled PROBLEM).

TIMSS PROBLEM 64
Arnold and Betty are farmers. Here is a map of part of their land. $\sphericalangle$ They want to straighten out the boundary between their properties without changing the amounts of land they each have. How can they do it?



TURN ONE INTO FOUR 68
Show that every triangle is a 4-reptile by demonstrating how they can be cut into four congruent triangles that are similar to the original triangle.

MYSTERY FUNCTION PROBLEM 90
A function, F, has only real numbers in its domain (its inputs), and its range (outputs) also consists of real numbers. For any real numbers a and b, $F(a + b) = F(a) + F(b)$. That is, if you put in the sum of two numbers, what comes out is the sum of the outputs for the two numbers fed in separately. What does F look like, algebraically?

For example, if $F(7) = 35$ *and* $F(9) = 45$*, then* $F(16)$ *would be* 80.

THE MEASURING CUPS PROBLEM 106
Kathryn has a cooking pot and two measuring cups. One cup holds 4 fluid ounces, the other holds 6 fluid ounces. Neither cup has marks that allow Kathryn to measure less than these amounts. Can she measure 2 fluid ounces using these cups? Can she measure 14 fluid ounces? 7 fluid ounces? For each amount she could measure, explain how.

RATIONAL POINTS ON THE UNIT CIRCLE 114
There are four integer-valued points on the unit circle; namely $(1, 0)$, $(0, 1)$, $(0, -1)$, and $(-1, 0)$. Are there any other *rational* points on the unit circle? If so, find at least six rational points in the first quadrant that lie on the unit circle.

THE PYTHAGOREAN THEOREM 115
Carefully state the Pythagorean Theorem. Be sure that your statement of the theorem will be clear to *anyone*, even if they have never heard of the theorem.

HOW ABOUT THE M-GRAPH? 117
The M-graph for $\overline{AB}$ is the set of all points M which are midpoints of $\overline{AC}$ where $\triangle ABC$ is a right triangle. What does the M-graph look like for a given $\overline{AB}$? Carefully explain your conclusions and reasoning.

EUCLID'S PROOF OF THE PYTHAGOREAN THEOREM 123
The following paragraph (and figure) provides the gist of the argument Euclid used in *The Elements* to prove the Pythagorean Theorem (Proposition 47 from Book 1). Fill in the details.

$Pas(n,k)$ denotes entry k in row n of Pascal's Triangle.