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# Introduction to Balanced Assessment

Assessing the mathematical performance of our students and the effectiveness of our mathematics instructional programs has become a major concern of the mathematics education community and the country as a whole. The National Council of Teachers of Mathematics (NCTM, 1995) has addressed that concern in its *Assessment Standards for School Mathematics*, which provides a set of six standards to guide the development of assessment instruments. This NCTM document makes clear, however, that it is a guide and not a "how-to" document. Guides are necessary but not sufficient, and teachers need different models of assessment that implement in concrete ways the principles set down in guidelines such as those offered by the NCTM.

The tasks in this collection were developed by the Balanced Assessment in Mathematics Program at the Harvard Graduate School of Education. The main goal of Balanced Assessment is to design assessments that can be used in classrooms throughout the nation—assessments that reflect the values of the mathematics reform movement as articulated in both NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics* and NCTM's (2006) *Curriculum Focal Points.* The assessments are intended to provide teachers, students, schools, and parents with useful information about how students and programs are doing with respect to those standards. The collection is balanced in regard to the mathematical content covered and the problem-solving skills utilized. The assessments have been created with a unique design philosophy that gets to the heart of the question "What is the structure of mathematics?"

## WHAT IS MATHEMATICS ABOUT? THE STRUCTURE OF THE SUBJECT

As in many disciplines, it is possible to identify both content and process dimensions in the subject of mathematics. Unlike many subjects where most of the process dimension refers to general reasoning, problem-formulating, and problem-solving skills, the process dimension in mathematics refers to many skills that are mathematics-specific. As a result, many people tend to lump content and process together when speaking about mathematics, calling it all mathematics *content*.

It is important to maintain the distinction between content and process. We say this because we believe that this distinction reflects something very deep about the way humans approach mental activity of all sorts. All human languages have grammatical structures that distinguish between noun phrases and verb phrases. They use these structures to express the distinction between objects and the actions carried out by or on these objects.

The content-process distinction in mathematics is best described by the words *object* and *action*. What are the mathematical objects we wish to deal with? What are the mathematical actions that we carry out with these objects? We will try to answer these questions in a way that makes clear the continuity of the subject from the earliest grades through post-secondary mathematics. Since there are really very few discrete categories of mathematical objects and actions, we believe this approach offers a simple and clear way to view and organize the teaching and learning of mathematics, consistent with NCTM guidelines for both teaching and assessment standards.

## THE OBJECTS OF MATHEMATICS

The first category of mathematical objects we consider is that of **number and quantity**. Indeed, elementary mathematics is largely about this group of objects and the actions we carry out with and on them. For this reason, a preponderance of the tasks in this collection are concerned with this topic. Some of the math objects included in number and quantity are

- integers (positive and negative whole numbers and zero);
- rationals (fractions, decimals, and all the integers);
- measures (length, area, volume, time, weight);
- real numbers (p, e, and all the rationals);
- complex numbers;
- vectors and matrices.

Along with number and quantity, we introduce very early a concern for another category of mathematical object, namely **shape and space**. Math objects investigated in this domain are

- topological spaces (concepts of connectedness and enclosure);
- metric spaces (with such shapes as lines/segments, polygons, circles, conic sections, etc.).

From the beginning we try to make students aware of **patterns** in the worlds of number and shape. In primary and elementary grades, patterns and sequences are closely aligned with arrangements. Pattern as a mathematical object matures into **function**, which is the central mathematical object of the subjects we call algebra and calculus. We include the following as math objects within the category **pattern and function**

- functions on real numbers (linear, quadratic, power, rational, periodic, transcendental);
- functions on shapes.

There are several other kinds of mathematical objects that have less prominent roles in the mathematics we expect our younger students to study. These include objects in the categories **chance and data** and **arrangement**.

The category **chance and data** is concerned with math objects such as

- relative frequency and probability;
- discrete and continuous data.

Some aspects of data collection, organization, and presentation can be done in the earliest grades, but little, if any, data analysis is performed. Notions of probability are not realistically addressable until late middle school.

In the earlier grades, **arrangement** tends to blend with the study of patterns of numbers and shapes. Some of the math objects to be considered in later elementary mathematics are

- permutations and combinations;
- graphs;
- networks, trees, and counting schemes.

## THE ACTIONS OF MATHEMATICS

As previously mentioned, the process dimension of mathematics has many actions that are mathematics-specific. It also involves actions that are properly regarded as general problem-formulating, problem-solving, and reasoning skills—processes that are needed across all aspects of learning and living. We divide these skills into four categories:

- Modeling/Formulating
- Transforming/Manipulating
- Inferring/Drawing Conclusions
- Communicating

With the exception of communicating, each of these actions has aspects that are specific to mathematics and aspects that are not specific to mathematics, but that are quite general in nature. Some of these general and specific aspects are now listed.

<u>Modeling/Formulating</u>

*domain-general*

observation and evidence gathering
necessary and/but not sufficient conditions
analogy and contrast

deciding, with awareness, what is important and what can be ignored

*domain-specific*

deciding, with awareness, what can be mathematized and then doing so

formally expressing dependencies, relationships, and constraints

<u>Transforming/Manipulating</u>

*domain-general*

understanding "the rules of the game"

understanding the nature of equivalence and identity

*domain-specific*

arithmetic computation

symbolic manipulation in algebra and calculus

formal proofs in geometry

<u>Inferring/Drawing Conclusions</u>

*domain-general*

shifting point of view

testing conjectures

*domain-specific*

investigation of limiting cases

investigation of symmetry and invariance

investigation of "between-ness"

<u>Communicating</u>

making a clear argument, both orally and in writing, using both prose and images

It is evident that there is no reasonable way to separate, nor should there be any interest in separating, the domain-specific and the domain-general aspects. We therefore come to the conclusion that it is better to parse the domain of mathematics as

**object** (number and quantity, shape and space, pattern and function, chance and data, arrangement)

and

**action** (including both domain-specific and domain-general actions)

rather than as

**content** (usually defined by "topics"—an undifferentiated mixture of objects and domain-specific actions)

and

**process** (i.e., domain-general actions), which is the usual procedure in mathematics education.

In transitioning to the objects × actions lens, it is helpful to begin by seeing the correspondence between the Balanced Assessment math objects and NCTM curriculum content. As shown in the following matrix, we can use the **objects × actions** structure to map the tasks and rubrics in this collection to NCTM's (2006, pp. 11–17) *Curriculum Focal Points*:

**Figure 1.1**

| **MAPPING OF BALANCED ASSESSMENT (BA) OBJECTS × ACTIONS<br>to<br>2006 NCTM CURRICULUM FOCAL POINTS** | |
|---|---|
| *CONTENT AREA/ MATH OBJECTS* | *LEARNING EXPECTATIONS* |
| NCTM:<br>Number and Operations, Measurement | NCTM:<br>Develop an understanding of whole numbers, including concepts of correspondence, counting, cardinality, and comparison. Identify measurable attributes and compare objects by using these attributes. (Pre–K)<br>Represent, compare, and order whole numbers, and join and separate sets. Order objects by measurable attributes. (K)<br>Develop understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts; also, whole number relationships, including grouping in tens and ones. (Gr. 1)<br>Develop an understanding of the base-ten numeration system and place-value concepts; develop quick recall of addition facts and related subtraction facts and fluency with multi-digit addition and subtraction. Develop understanding of linear measurement and facility in measuring lengths. (Gr. 2)<br>Develop understandings of multiplication and division and strategies for basic multiplication facts and related division facts; also fractions and fraction equivalence. (Gr. 3)<br>Develop quick recall of multiplication facts and related division facts and fluency with whole number multiplication; also decimals, including the connections between fractions and decimals. Develop an understanding of area and determine the area of two-dimensional shapes. (Gr. 4)<br>Develop an understanding of and fluency with division of whole numbers; also addition and subtraction of fractions and decimals. Describe three-dimensional shapes and analyze their properties, including volume and surface area. (Gr. 5) |

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**Figure 1.1**
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<table>
<tr><th>CONTENT AREA/ MATH OBJECTS</th><th>LEARNING EXPECTATIONS</th></tr>
<tr><td>BA:<br>Number and Quantity</td><td>BA:<br><u>Primary Grades</u>: Students will grow in their capacity to<ul><li>demonstrate a robust understanding of our numeration system;</li><li>demonstrate understanding of the conceptual meaning of addition and subtraction of whole numbers and integers, and the relationship between the two processes;</li><li>demonstrate a basic understanding of the meaning of simple fractional parts;</li><li>count anything in the world around us, and identify and measure continuous quantities such as length, area, and time.</li></ul><u>Elementary Grades</u>: Students will grow in their capacity to<ul><li>demonstrate computational facility with the four arithmetic operations on whole numbers and integers;</li><li>demonstrate understanding of the various meanings of multiplication and division of whole numbers and integers, and the relationship between the two processes;</li><li>make reasonable approximations of the results of arithmetic computations and estimates of measurement using standard and non-standard measures;</li><li>demonstrate understanding of the order properties of decimals and other rational fractions.</li></ul></td></tr>
<tr><td>NCTM: Algebra</td><td>NCTM:<br>Recognize and duplicate simple sequential patterns. (Pre–K)<br>Identify, duplicate, and extend simple number patterns and sequential and growing patterns as preparation for creating rules that describe relationships. (K)<br>Identify, describe, and apply number patterns and properties. (Gr. 1)<br>Use number patterns to extend knowledge of properties of numbers and operations. (Gr. 2)<br>Create and analyze patterns and relationships involving multiplication and division. (Gr. 3)<br>Develop an understanding of the use of a rule to describe a sequence of numbers or objects. (Gr. 4)<br>Use patterns, models, and relationships as contexts for writing and solving simple equations and inequalities; create graphs of simple equations; develop an understanding of the order of operations. (Gr. 5)</td></tr>
</table>

**Figure 1.1**
Continued.

| *CONTENT AREA/ MATH OBJECTS* | *LEARNING EXPECTATIONS* |
|---|---|
| BA:<br><br>Pattern and Function | BA:<br><br>Primary Grades: Students will grow in their capacity to<br>• recognize and extend numerical patterns or relationships;<br>• recognize and extend spatial patterns;<br>• enumerate and organize simple arrangements.<br><br>Elementary Grades: Students will grow in their capacity to<br>• express, both in words and symbolically, how one thing depends on another;<br>• identify, generate, extend, and describe repetitive relationships, both numerical and spatial. |
| NCTM:<br>Geometry | NCTM:<br><br>Identify shapes and describe spatial relationships. (Pre–K)<br><br>Identify, name, and describe a variety of shapes and spaces. (K)<br><br>Compose and decompose geometric shapes. (Gr. 1)<br><br>Estimate, measure, and compute lengths while solving problems involving data, space, and movement through space. (Gr. 2)<br><br>Describe and analyze properties of two-dimensional shapes; introduce<br><br>concepts of symmetry and congruence. (Gr. 3)<br><br>Deepen and extend understanding of two-dimensional space by finding area; use transformations to design and analyze simple tilings and tessellations. (Gr. 4)<br><br>Describe three-dimensional shapes and analyze their properties, including volume and surface area. (Gr. 5) |
| BA:<br><br>Shape and Space | BA:<br><br>Primary Grades: Students will grow in their capacity to<br>• distinguish and name a variety of two-dimensional shapes;<br>• demonstrate understanding of the symmetries of shapes.<br><br>Elementary Grades: Students will grow in their capacity to<br>• distinguish, name, and manipulate a variety of two- and three-dimensional shapes;<br>• demonstrate understanding of the geometric concepts of distance, location, symmetry, similarity, translation, rotation, reflection, covering, projection, scaling, and tessellation. |

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**Figure 1.1**
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| *CONTENT AREA/ MATH OBJECTS* | *LEARNING EXPECTATIONS* |
|---|---|
| NCTM:<br>Data Analysis, Probability | NCTM:<br>Use the attributes of objects (size, quantity, orientation, color, etc.) to describe, sort, and compare. (Pre–K)<br>Sort objects and use one or more attributes to solve problems. (K)<br>Solve problems involving measurements and data; represent discrete data in picture and bar graphs. (Gr. 1)<br>Construct and analyze frequency tables, bar graphs, picture graphs, and line plots, and use them to solve problems. (Gr. 3)<br>Apply understanding of place value to develop and use stem-and-leaf plots. (Gr. 4)<br>Apply understanding of whole numbers, fractions, and decimals to construct and analyze double-bar and line graphs; use ordered pairs on a coordinate grid. (Gr. 5) |
| BA:<br>Chance and Data, Arrangement | BA:<br>Primary Grades: Students will grow in their capacity to<br>• collect, organize, and display simple data sets;<br>• make decisions based on provided data.<br>Elementary Grades: Students will grow in their capacity to<br>• organize and display discrete information in a variety of formats, including frequency tables, bar graphs, and line plots;<br>• demonstrate competence in dealing with the effects of randomness and uncertainty in data collection;<br>• enumerate and organize simple combinations and permutations. |

## WEIGHTING OF TASKS

In order to approach the problem of designing balanced assessment packages in mathematics, one must have a clear view of the kind of understanding and the skills that we wish to assess in our students, and the ways in which the tasks we design elicit demonstrable evidence of these skills and understanding. In what follows, we shall describe how our view of the subject of mathematics—its objects and its actions—informs the design of tasks and the balancing of assessment packages.

Each task is classified according to domain, that is the mathematical objects that are prominent in the accomplishment of the task. Most of our tasks deal predominantly with a single sort of mathematical object, although some deal with two. Each task offers students an opportunity to demonstrate a variety of kinds of skill and understanding.

In order to score student performance on a task, one has to first analyze the task and decide on the nature of the demands that the task makes on the student. We considered the following four kinds of skill and understanding:

*Modeling/Formulating*: How well does the student take the presenting statement and formulate the mathematical problem to be solved? Some tasks make minimal demands along these lines. For example, a problem that asks students to calculate the length of the hypotenuse of a right triangle given the lengths of the two legs does not make serious demands along these lines. On the other hand, the problem of how many 3-inch diameter tennis balls can fit in a 3-inch × 4-inch × 10-inch rectangular or parallelepiped box, while exercising the same Pythagorean muscles in the solution, is rather different in the demands it makes on a student's ability to formulate problems.

*Transforming/Manipulating:* How well does the student manipulate the mathematical formalism in which the problem is expressed? This may mean adding two numbers or dividing one fraction by another, making a geometric construction, solving an equation or inequality, plotting graphs, or finding the derivative of a function. Most tasks will make some demands along these lines. Indeed, most traditional mathematics assessment consists of problems whose demands are primarily of this sort.

*Inferring/Drawing Conclusions:* How well does the student apply the results of his or her manipulation of the formalism to the problem situation that spawned the problem? Traditional assessments often pose problems that make little demand of this sort. For example, students may well be asked to demonstrate that they can multiply the polynomials $(x + 1)$ and $(x - 1)$, but may not be expected to notice (or understand) that the numbers one cell away from the main diagonal of a multiplication table always differ from perfect squares by exactly 1.

Similarly, a younger student might be asked to sort through what may seem to be contradictory pieces of information in order to develop a solution strategy.

*Communicating*: How well do students communicate to others what they have done in formulating the problem, manipulating the formalism, and drawing conclusions about the implications of their results?

Since we do not expect each task to make the same kinds of demands on students in each of the four skill/understanding areas, we assign a single-digit measure of the prominence of that skill or understanding area in the problem according to the following scale of weighting codes. ***Note that these numbers are not measures of student performance, but are measures of the demands of the task for a given performance action. You might think of this as akin to an Olympic dive or figure skating maneuver being given a numerical rating.***

**WEIGHTING CODES:**
0 = not present at all
1 = present in small measure
2 = present in moderate measure, and affects solution
3 = a prominent presence
4 = a dominant presence

## USING THE TASKS

These assessments can be used in a variety of ways, depending on your local needs and circumstances.

- At each grade level, the balanced assessment tasks provide opportunity for integrated, classroom-based formative assessment. The collection allows you to select tasks that are appropriate at particular points in the curriculum or that specifically address a mathematics action that students need help with. As previously mentioned, the collection is balanced as to both content (math objects) and process (math actions), and is developmentally appropriate for the particular age group. Using the tasks as formative assessment enables the teacher not only to adjust instructional strategy for the whole class, but also to pinpoint individual weaknesses.
- At each grade level, the balanced assessment tasks can be used as exemplars for open-response questions on high-stakes tests. They provide students with the opportunity to work on organizing their mathematical thinking and to practice and refine their communication skills.
- At each grade level, the balanced assessment tasks can serve as a transition toward a standards-based curriculum or as enrichment for existing curriculum. They are designed to be used as a supplement to any standards-based curriculum.
- At each grade level, the balanced assessment tasks can be used as pretest/posttest items for diagnostic purposes. They may also be used as summative assessment for specific topics or for additional information when assigning a mathematics grade.

It is extremely important that you work through a task yourself before giving it to your students. Only in this way can you become familiar with the context and the mathematical demands and be able to anticipate what needs to be highlighted as you launch the task. At the beginning of each task, you will find a teacher's guide that details

- the math object category (some tasks fall into more than one category; the dominant object is indicated by a bold **X**);
- the process, or math actions, weightings;
- assumed mathematical background;
- core elements of performance ;
- specific directions for launching and conducting the task;
- possible extensions;
- materials (paper and pencil are assumed to be necessary for all tasks; in some cases, additional materials and calculators are stipulated).

A pre-activity is provided for many of the tasks. It is expected that the teacher will do the pre-activity with the whole class and answer any questions before assigning the main body of the task.

Depending on whether the task is being used as formative or summative assessment, the launch of a task will vary. In some cases, it will be necessary only to distribute the task to students and let them read and work through it on their own. In other cases, it may be more productive to have them work in pairs, but report back individually. When students are meeting this type of task for the first time, especially when the tasks are being used primarily as learning tasks to enhance the curriculum, you may decide to work through the tasks item by item, talking with the students and posing questions when things get "stuck." This type of informal assessment gives you the opportunity to observe what strategies students favor, what kinds of questions they ask, what they seem to understand and what they are struggling with, and what kinds of prompts get them "unstuck."

Younger students will need assistance in the scribing of their answers—it is very common for verbal exposition to develop before writing ability. By the middle of first grade, however, students should be encouraged to put pencil to paper in order to chronicle their mathematical activity.

We cannot stress enough the importance of teachers working through the entire task completely before using it in the classroom; it is only in this way that teachers can anticipate where their students may run into difficulty. It is also imperative that teachers are aware of, and are comfortable with, all possible solutions—in other words, there is often more than one "right" answer or approach.

## USING THE RUBRICS

Rubrics are a set of rules or guidelines for giving scores to student work; they answer the question "What do the varying degrees of mastery for this task look like?" The rubrics that accompany these assessments are based on the Core Elements of Performance that are identified for each task and may be used in a variety of ways. If the tasks are being used as formative assessment, students should be allowed to revise their work to meet as closely as possible the criteria for "full competency." If the tasks are being used as summative assessment, the partial- and full-competency descriptors can be restated as a four-level holistic rubric. Level 4 work meets all the descriptors for "full competency"; Levels 1–3 are arrived at by appropriately adjusting the descriptors for "partial competency."

Teachers may need to translate scores from balanced assessment tasks to a letter- or number-grade system. While it may not be possible to preserve all four aspects of the mathematics actions, these scores should be aggregated no further than a separate score for skills (Transforming/Manipulating) and a separate score for understanding, which combines both Modeling and Inferring. Collapsing the individual scores substantially reduces the utility of these materials to provide a mathematical profile of student understanding and to contribute to making informed decisions about students. Information on using these rubrics as part of a complete scoring system with accompanying software can be found in the document *MCAPS: Mathematics Content and Process Scoring* (Schwartz & Kenney, 1999).

These rubrics can also be used to facilitate professional development activities for teachers, either in a study group that focuses on scoring student work or an action-research project on formative assessment. The conversations that ensue when teachers look at their own students' work, compare it against the criteria set forth in the rubric, and then discuss it with their colleagues, provide a clear perspective as to where students had difficulties and where they were successful. This, in turn, leads to better pedagogy and enhanced mathematical understanding for both teacher and student.

## SUMMARY

Any assessment that is truly worthwhile to teachers, students, and others with a valid interest in what students can do mathematically must have the following characteristics, as indicated in NCTM's (1995) *Assessment Standards*:

- **The assessment must focus on important, grade-level-appropriate mathematics.** Since it is not possible to assess everything that students have learned, it is important to select carefully what learning is assessed by concentrating on the most important and useful mathematics taught and learned at that grade level.
- **The assessments must be worthwhile learning activities—not digressions from learning.** For the student, assessment is a tool that helps further the understanding of important mathematical ideas. For the teacher, assessment is student work that informs and augments instruction. Worthwhile assessment is not something students and teachers "stop and do," but a way to further what they are already doing.
- **The assessments must maintain a focus on accessibility and equity for all students.** The student must have, and the teacher and student must perceive that the student has, a fair opportunity to do his or her best. Assessments are designed to provide a student of either gender and of any cultural, linguistic, and socio-

economic background with the means to do his or her strongest mathematical work.

- **The assessments must elicit scorable, informative student work.** The assessments are designed to result in more than just an answer from the student. Rather, students are asked to solve a problem, show their thinking, and create a product. The information in the student's response, and the features of the student's work that are evaluated, give a picture of his or her understanding of mathematical concepts, strategies, tools, and procedures.

It is the hope and intent of the authors that the balanced assessment tasks and rubrics provided in the following chapters will assist teachers in their efforts to put into practice the theoretical guidelines provided by NCTM for the assessment of learning in mathematics. They are designed with great care to make them as revealing and adaptable as possible, suitable for incorporation into any curriculum, and a source of important information about students' mathematical understanding. We hope that teachers will also find within these assessments new ways to think about teaching mathematics, new ways to sustain dialogue with students, and new impetus for conversation with colleagues about the important work in which teachers engage.