1

Equality

Paul Parslow-Williams and Anne D. Cockburn

Pause for thought

What do you think of when you see this sign '='? If you had to discuss it with a class of children, what would you say? Would you be tempted to describe it as a symbol for 'makes'? How about 'the same as'? Do you always write number sentences with this sign at the end?

When we had a look at what people have been writing about the equals sign (see, for example, Behr *et al.,* 1980; Falkner *et a*l., 1999; Freiman and Lee, 2004) we discovered it is well documented that children frequently find it difficult to appreciate that:

- '=' signifies 'the same as', but not necessarily 'identical to'.
- The equals sign is not a request to *do* something: '+' invites you to add items, '–' asks you to subtract but '=' simply states the situation rather than demanding any action.

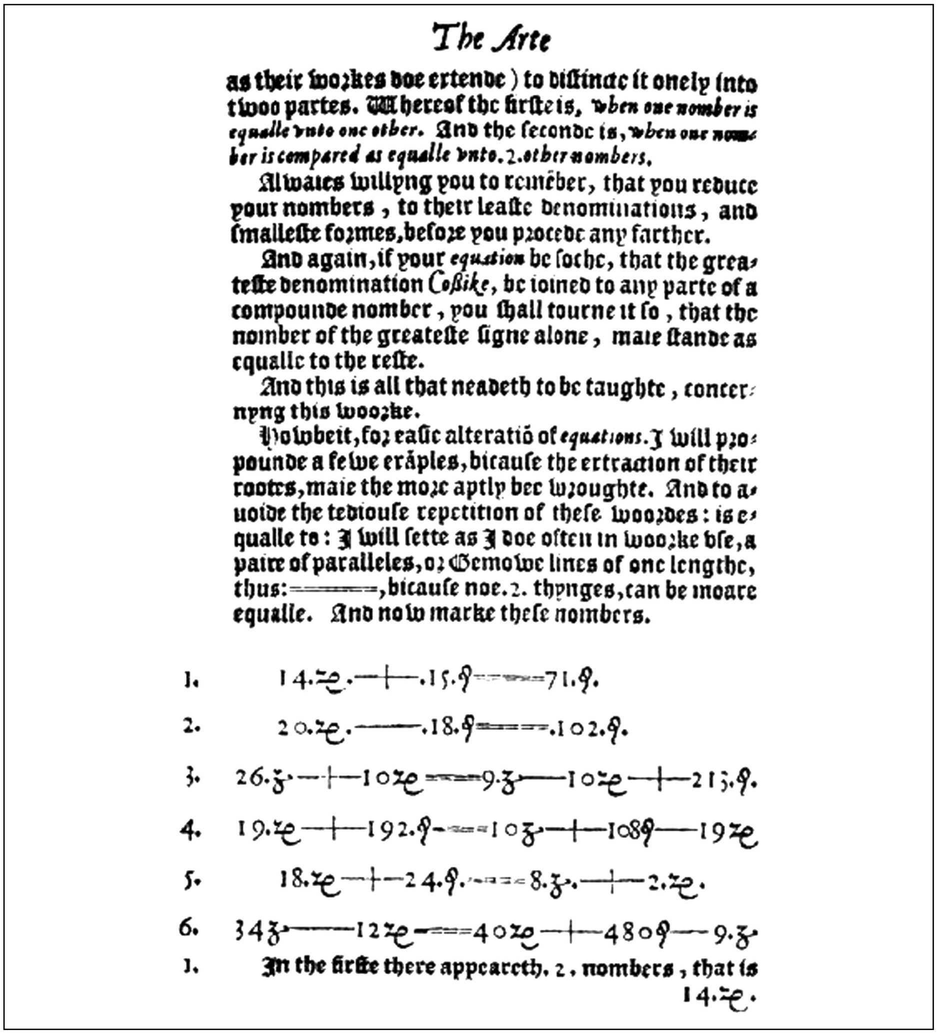
Jones (2006: 6) puts it: 'An arithmetic expression is like a film set on which the numbers are actors, the operators are the script and the equals sign the director who shouts "Action!"' If you are an experienced teacher, you were probably not surprised to read this, but we hope that, like us, you feel that simply knowing the misconceptions children hold is not enough and are intrigued to delve further into why children experience these particular difficulties. To this end, in this short chapter we will investigate some of the most common misconceptions we observed around the equals sign. They will illustrate the above but, more significantly for our purposes, they provide insight into how children try to make sense of the world of mathematics and how we might better help them to do so. Much of the material discussed arises from children's responses to the equality problems presented in Figures 1.2 and 1.3. Before we turn to these, however, let us take a few steps back in time.

A brief history of the equality symbol

Whilst we may take the equality symbol '=' for granted as part of our everyday mathematical vocabulary, it was not until the sixteenth century (1557) that this form, albeit in a rather elongated version, was first seen in print in Robert Recorde's *Whetstone of Witte* (a whetstone is a device for sharpening tools and, in the title of Recorde's book, it is assumed that the 'witte' being honed is one's mathematic understanding). Prior to this, equality had been symbolised in a variety of ways, including the 't' used in the third-century manuscripts of Diophantus, '*ae*' (an abbreviation for the Latin word *aequalis*) and a pair of vertical lines '||'(Cajori, 1928; Saenaz-Ludlow and Walgamuth, 1998).

Recorde (1557) rather elegantly justifies its use, explaining (see Figure 1.1):

Figure 1.1 Extract from Robert Recorde's *The Whetstone of Witte*, 1557 (Cajori, 1928: 165))



The Arte

as their workes doe extende) to distincte it onely into twoo partes. Whereof the firste is, *when one nomber is equalle unto one other.* And the seconde is, *when one nomber is compared as equalle unto .2. other nombers.*

Alwaies willyng you to remēber, that you reduce your nombers, to their leaste denominations, and smalleste formes, before you procede any farther.

And again, if your *equation* be soche, that the greateste denomination *Cossike*, be ioined to any parte of a compounde nomber, you shall tourne it so, that the nomber of the greateste signe alone, maie stande as equalle to the reste.

And this is all that neadeth to be taughte, concernyng this worke.

Howbeit, for easie alteratiō of *equations*. I will propounde a fewe exāples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe .2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ze. + .15.ϕ = 71.ϕ.
2. 20.ze. − .18.ϕ = .102.ϕ.
3. 26.ʒ. + 10ze = 9.ʒ − 10ze + 213.ϕ.
4. 19.ze + 192.ϕ = 10ʒ + 108ϕ − 19ze
5. 18.ze + 24.ϕ. = 8.ʒ. + 2.ze.
6. 34ʒ − 12ze = 40ze + 480ϕ − 9.ʒ.

1. In the firste there appeareth .2. nombers, that is 14.ze.

Figure 1.2 A series of equality problems based on addition

Can you complete these number sentences?

a) $7 + 2 = \square$

b) $5 + \square = 8$

c) $\square + 4 = 9$

d) $\square = 3 + 4$

e) $5 = \square + 1$

f) $8 = 5 + \square$

g) $5 + 4 = \square + 8$

h) $6 + 2 = 3 + \square$

i) $1 + \square = 6 + 2$

j) $\square + 3 = 7 + 2$

k) $5 + \square = \square + 7$

l) $9 = \square$

m) $5 + 4 = \square + 6 = \square$

n) $4 + 3 = 2 + \square = \square + 1 = \square$

> To auoide the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe* lines of one lenghte, thus: ═════, bicause noe.2. thynges, can be moare equalle.

Despite his significant contributions to mathematics, which also include the introduction of algebra to the British Isles and authorship of a series of mathematical texts in the English language making geometry and astronomy accessible to wider audience, Recorde is relatively unknown. If you would like to find out more about this fascinating Welshman, including his rise and fall in Tudor politics and how he came to a rather unfortunate end as a pauper in jail, there are a number of short biographies to be found on the Internet (see, for example, his entry in the online Encyclopædia Britannica). If you wish to focus on his mathematical accomplishments in a wider context, you will enjoy reading Cajori (1928).

However, before we move on, it is worth noting that although Recorde is usually credited with being the first to use this particular symbol, there is some evidence in the form of a manuscript from the University of Bologna that suggests that it may also have been developed elsewhere (Marchini, personal communication, 2007).

Challenge

Ask your class – regardless of their age – to complete Figures 1.2 and 1.3.

---

* 'Gemowe' means 'twin'.

Figure 1.3 A series of equality problems based on subtraction

Can you complete these number sentences?

a) 9 – 3 = □

b) 7 – □ = 2

c) □ – 4 = 5

d) □ = 4 – 1

e) 5 = □ – 1

f) 5 = 7 – □

g) 5 – 4 = □ – 8

h) 6 – 2 = 9 – □

i) 7 – □ = 8 – 2

j) □ – 3 = 7 – 5

k) 6 – □ = 8 – □

l) 5 – 4 = 7 – □ = □

m) 8 – 5 = 5 – □ = 6 – □ = □

Ivan, a year 1 teacher, presented Figures 1.2 and 1.3 as ‘problems of the week’ and invited the children to ‘have a go’ and to stop when they wished. He noted that children were more likely to attempt the problems involving the ‘+’ sign as opposed to those including the ‘–’ sign. Furthermore, their success rate was higher for addition than subtraction. From this he concluded that the children were ‘obviously more confident in their own adding ability than subtracting’. Ivan may be correct that this was his pupils’ perception but, interestingly, to solve a tricky problem such as 5 = □ + 1 it is highly likely that the children would have used one of the following strategies:

- a ‘trial and improvement’ approach that involved inserting different numbers in the place of the unknown until balance was achieved, possibly thinking along the lines of ‘What do I need to add to 1 to make 5?’ or
- subtraction.

Similarly, almost all of the class successfully solved 5 + □ = 8, for which they almost certainly used ‘counting on’ to 8. The initial point we wish to make here is that, although the use of a particular sign (in this case ‘+’) may have given the children more confidence as Ivan suggested, it did not necessarily mean that they were performing the simple operation one instinctively associates with the '+' sign.

The second point from the above is to question the notion that Ivan's class saw '=' as a command for action which, coupled with '+' or '–' in these cases, was to add or subtract. Certainly a couple of children did respond 5 = $\boxed{6}$ + 1 but none of them answered 5 + $\boxed{13}$ = 8. Further support for this idea comes from the observation that, although over half of them successfully inserted '9' in □ – 4 = 5, only one child responded '1'. Comparing the above with the responses of Italian children of similar age, it is interesting to note that over 90 per cent answered 5 + □ = 8 correctly, implying that also for them '=' together with '+' does not necessarily involve the need in children's minds to add 5 and 8.

Looking at the responses to the more complex problems in Figures 1.2 and 1.3 provides further insight into children's perceptions of '='. Take 5 + 4 = □ + 8 for example. Half of the year 1 children correctly responded '1', but a quarter of them inserted '9' (a comparable proportion of Italian children responded similarly). If we also consider 5 – 4 = □ – 8, we find that only one of the 22 children in Ivan's class was correct while almost half of them gave the answer as '1'. Similarly, many Italian pupils in years 2, 3 and 4 also responded with '1'. Ivan explained one of the most likely reasons for these responses in terms of Adam's answer:

> He has put 1 when it should be 9. I think he has seen the bit he knows, i.e. 5 – 4 = 1, because that is the bit he could do and there is this perception that the answer always comes after the equals sign (he ignored the –8). When I spoke to him about it he didn't understand it.

It may be that, when uncertain, children will tackle what they are familiar with in an endeavour to do as much of the work as they can for you. Theo (aged 6), for example, produced the following: 7 – 13 = 8 – 2. In effect he took 2 from 8 and arrived at 6 and then worked out what needed to go into the box to produce a balance of 6 on the left-hand side of the equation. The idea of balance is taken up again later in the chapter.

You may, like some of the teachers in the project, feel that 9 = ? is a rather unconventional question to present to children, yet many of the responses we collected from pupils of all ages were illuminating. Thus, for example, Renato, a year 2 child in Italy, believing that the equals sign should always be coupled with an operator such as '+' or '⊗', wrote 9 = 0 when confronted with 9 = □, explaining 'I wrote zero because there is nothing to do with the 9'. A year 6 child produced an equally unexpected but valid response, writing 9 = 27 and then dividing the response box into two to indicate '2' and '7' as separate entities. His teacher suggested that:

> Many (even Level 5) struggled with 9 = □ or 42 = □. Pupils are so accustomed to having to do something.

None of the children in Ivan's class showed any of their workings when attempting to solve the problems. However, when Ruth asked seven of her

Figure 1.4 Examples of pupils' recording strategies for solving equality problems

6 – 2 = 9 – ?

////XX ////XXXXX

? – 3 = 7 – 5

//XXX //XXXXX

5 = ? – 1

////X

pupils to complete the subtraction problems they all included tally marks to indicate their thinking. As Ruth explains:

> They've been encouraged from day 1 to show their working. They've been drilled into it. They are so used to using structural apparatus that when there are no cubes available they do sticks and dots.

Their strategy was typically to set out sticks underneath on either side of the equals sign and score out the sticks to be subtracted (see Figure 1.4). Ruth's children – at the age of 6 – were a year older than Ivan's, but it is interesting to compare their responses in the light of their differing approaches. When confronted with the problems in Figure 1.4, the recording strategy initially proved productive for Ruth's class, who were able to answer the first two questions correctly, whereas few of Ivan's children found the correct answers. In the first, the left-hand side provided a pattern, so the pupil had only to make the nine strokes and cross out sticks one at a time until they were left with a similar pattern of four uncrossed sticks. The middle example arguably demanded a much higher order of thinking than the other two; whilst the right-hand side gave the pattern, it was not immediately clear to the pupils how many sticks to draw. To reason that three crossed sticks of a yet unknown number must be placed and then to put two uncrossed sticks on the end (to match those on the other side) was quite a difficult process. But note that none of those who did workings succeeded with the third question. All the year 2 children drew five tallies (/////) and crossed out one (///X). It was not uncommon in the project to find that children experienced problems when attempting to apply naïve strategies to situations that differed from the context in which they were introduced. Even a small change, such as the number of terms in the question, was often sufficient to confuse many of the children. For a theoretical framework and practical advice on supporting pupils with developing their mathematical strategies, please refer to Chapter 6.

Challenge

Consider how you might complete the following: 6 ? 1 ? 5 (each ? represents a mathematical symbol). Is there more than one possible solution?

As we read through the children's responses we frequently gained the impression that they were trying to make sense of what they had been asked to do. The most striking examples of children seemingly applying their own logic were revealed by problems such as □ = 3 + 4 and □ = 4 – 1, to which many children responded with the answers of 1 and 5, respectively. It is possible that these children, in effect, did the opposite operation to the one required and made us question whether, on seeing a problem written 'reversed', they assumed that their task was also reversed – to add rather than subtract and vice versa.

Alternatively, might the children have been ignoring the = and +/– signs in order to force a left to right operation? To take □ = 3 + 4, you could also arrive at 1 by thinking of the calculation as 1 '+' 3 '=' 4. This also holds true for the second example, since 5 will also be the result if □ = 4 – 1 is considered as □ – 4 = 1. In both of these cases, perhaps the position of the symbol is more important than what it represents. Another possibility is that the position of the symbol is irrelevant in the minds of some of the children but its function is crucial.

Challenges

What errors do your pupils make when doing problems involving the equals sign? Do you notice any pattern to their mistakes? What misconceptions might these reveal?

Contributory factors

Not to put too fine a point on it, the project teachers were amazed by their pupils' responses to our equality problems. As they explained, previously, they had not perceived the seemingly innocent '=' sign as an issue:

> We never discuss explicitly what this sign means and I feel that this is something that as a school we need to look at now. (Linda, year 2 teacher)
>
> We didn't talk about the equals sign and what it meant at teacher training. (Kath, year 6 teacher)
>
> We use the equals sign all the time but never talk about what it means. We don't make the connections when we look at it. (Laura, year 3 teacher)
>
> We teach the inequality signs but there is little emphasis on the equals sign. The children would probably find ? > 3 + 4 easier than ? = 3 + 4 (Donna, year 4 teacher)

Previously lacking an appreciation of the equals sign, Kath (year 6 teacher) confessed that she was probably responsible for some of her pupils' errors such as the example shown below:

Problem: 48 – □ = 47 – □ = 46 – □ = □

Pupil response: $48 - \boxed{1} = 47 - \boxed{1} = 46 - \boxed{1} = \boxed{45}$

It appears that the pupil interpreted the problem as a chain of commands leading to a final answer rather than one long number sentence with three equality symbols. Kath went on to explain that, in the past, she had regularly used the equals sign when it is not mathematically appropriate. Thus, for example, in order to demonstrate partitioning strategies to pupils, she might have written

$$45 + 22 = 40 + 20 = 60 + 5 = 65 + 2 = 67$$

The project made her

> think about how I write on the board. I should use arrows more. Today has been useful from my point of view as a teacher. I do lots of things subconsciously that are causing misconceptions. Now I'm going down to a younger year group I have a better understanding of what I need to be really careful about.

Seeing results from both his own and other classes, Ivan also concluded that some of his teaching strategies might have contributed to children developing misconceptions which, although not apparent in the short term, created difficulties at a later stage in their schooling. He reflected:

> You focus on your own group and forget what they are going on to. I've done year 3,2,1 and R and so you get an idea of what the children are moving towards … [but] some of the children have misconceptions and still have them 5 years on.

(See chapter 6 for actual examples of the same misconceptions in year 2 and year 10.) Ivan further explained:

> You tend to cover things very quickly because of the expected coverage when the children are not getting the bedrock. I have done 'if you add 2 things it gets bigger and if you subtract it gets smaller'. You don't tend to think what you do has bearing on what others do.

This mirrors findings in another research project and is reported in Cockburn (2007).

Challenge

What might you do which could contribute to your pupils misunderstanding of the equality?

Some ways forward

Clearly children have to develop a real appreciation of the meaning of mathematical equality which, traditionally, is represented by '='. This is not just to complete simple addition, subtraction, multiplication and division problems and not just to tackle the vaguely familiar subject of algebra which, to an early years teacher, may seem so far into a child's future as to be almost irrelevant. Indeed, reflecting on the challenge below, you will note that we expect children to have a fairly developed sense of both the composition of numbers and the relationships between them by the time they are engaging in what traditionally was termed 'mental arithmetic' in year 1.

Challenge

Reflect on what you have read in this chapter and the implications it might have for teaching the following relationships:

7 + 5 = (2 + 5) + 5 = 2 + (5 + 5) = 2 + 10 = 12
24 = 20 + 4…

Rather than viewing young children as 'empty vessels' who need to acquire a mass of skills before they can advance mathematically, we think it would be helpful to adopt a different perspective at this point and consider what they can do, rather than what they cannot.

For years, for example, it has been recognised that children as young as 3 years old can successfully complete a range of numerical problems – including division – if presented in an appropriate manner using real objects within a familiar everyday context (see, for example, Desforges and Desforges, 1980; Hughes, 1986; Gelman and Gallistel, 1978). Your own experience may also tell you that, when totting up the numbers on the register, Key Stage 1 children can solve problems which, when written formally, might be mathematically represented by 24 – □ = 21 or even 15 + □ = 21, if you were considering how many children had opted for packed lunches and the number of hot dinners needed to be calculated. We will return to this later, but first let us consider another typical aspect of young children's mathematical knowledge.

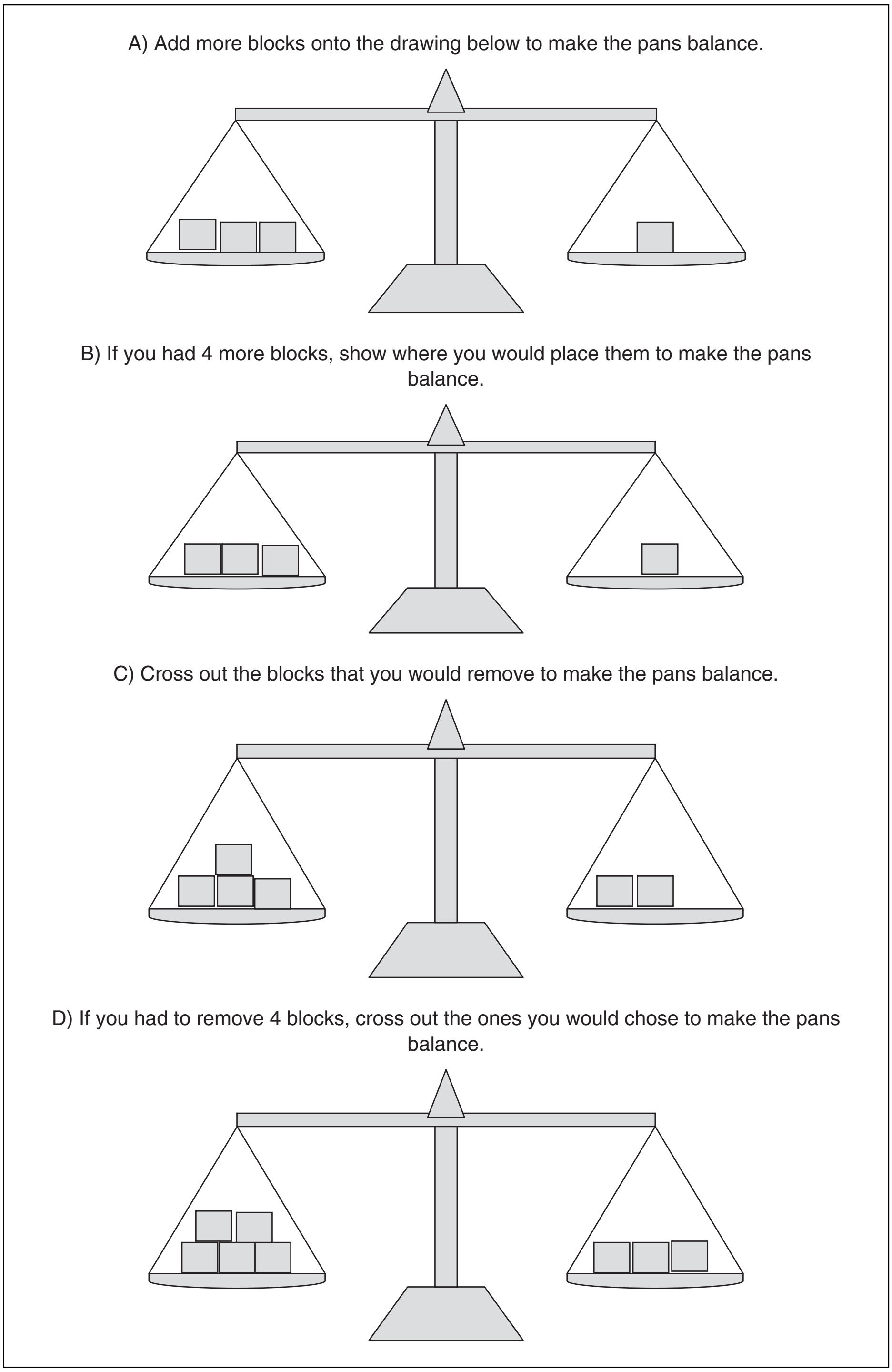
In addition to asking pupils to complete formal equality problems as presented in Figures 1.2 and 1.3, We asked them to undertake a series of tasks to explore their understanding of equality in a visual sense, as illustrated in Figure 1.5.

Sandra asked five of her reception class to complete the exercise. Three of the 5-year-olds successfully completed (A) independently by drawing two blocks on the right-hand pan, explaining:

> That's got 3 and that's got 3. (Kareen)
>
> Now it will balance. (Lorna)
>
> I drew 2 blocks to make it the same. (Ron)

Neil and Miranda also achieved success but adopted unexpected strategies. Neil began by drawing three cubes on the left-hand pan and then drew a further 5 on the right hand side and explained 'Six and six'. Entirely separately, Miranda drew 5 blocks on the left hand side and 7 on the left and announced, 'Eight on each side'.

Figure 1.5 Equality problems represented in a visual form



The children's responses to the other tasks were fairly standard, with their comments suggesting a good understanding of equality, such as 'Now there is 4 on each side' (Kareen) and 'I've made 2 and 2' (Ron). It is interesting to note though that, when completing (C), both Neil and Lorna – again independently – crossed out all of the cubes in sight. Viewing the result, Neil commented, 'Nothing on each side', and Lorna remarked 'Now there's none on each side so they [the pans] will be the same height'.

The children's teacher was not surprised, as she said that she has noticed over the years that reception children generally find conceptions surrounding weight straightforward and easy. She continued:

> They readily understood the balance so let's start to push their thinking out … It's building on previous experience which has been fairly well digested. It's giving them another hook into the learning: a visual hook. It's giving them another way in. Verbal language doesn't necessarily unlock doors for them.

A year 2 teacher from another school, Linda, described how she had in the past used:

> balances specifically to demonstrate equality. This is a very powerful method for introducing this concept. Three cubes on one side, one of the other. Pupils can see (since one side is lower) that these are not equal. We used to do lots of work on equality and talked about balancing using apparatus where appropriate, but we haven't done a lot of this since the introduction of the Strategy. Looking through these examples has made me think about my teaching of equality. When looking at the tasks I thought 'I don't do this sort of thing with my children'.

She went on to say, however, that her

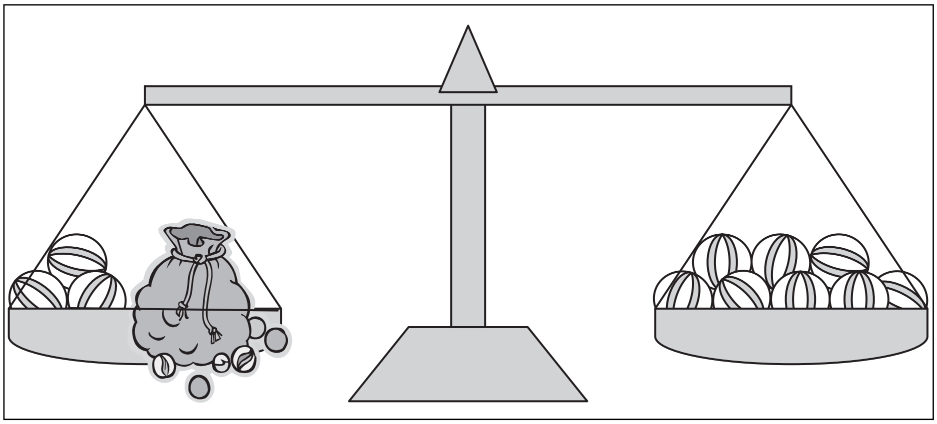
> pupils found it difficult when balances were on paper. When the children were asked to add four cubes (question (B)), they wanted to put the four cubes on one side. They didn't understand that they could split the set. The children expected the solutions to be simpler and wanted to solve them quickly and at a superficial level.

Kath – a year 6 teacher – also referred back to her previous practice when she saw the results of the task:

> I wish we still had the old equaliser balances since when we went back to showing them the balances, they found it much easier to understand … [at the moment] they understand equals as makes rather than a balance.

So, to return to young children's facility to solve realistic problems: have we been seduced into thinking that the transition to formally recording such situations is easier than it actually is? We know from the work of others that they can record their work informally using symbols and pictures (e.g. Hughes, 1986). We have observed that children as young as 5 understand the practical concept of balance and hence equality. Might the two approaches be combined so that pupils were encouraged to undertake their calculations and illustrate

Figure 1.6 Visual representation of a 'missing number' equality problem using a suspension balance



their thinking perhaps using, in the first instance, real objects such as marbles or miniature bananas and classroom balances?

Working with a top-pan or suspension balance offers children a concrete experience of equality and an opportunity to engage physically with number sentences. Thus, for example, the following conversation might take place:

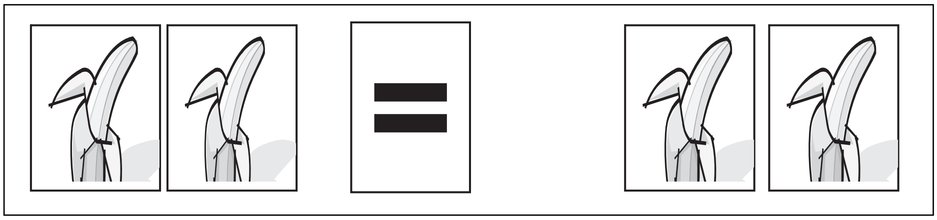
> Teacher: If I have 2 bananas and you have 3, how many would you need to eat so that you have the same number of bananas as I have? How could we show it on the balance?
>
> Sam: You could put your 2 bananas on your side and I could put my 3 on mine. That makes my side lower but if I take one off my side – I could pretend to eat it! – then it balances.

To take things a step further, drawstring bags could be used to represent unknown sets. For example, to demonstrate the problem $3 + \square = 8$, a bag containing five hidden marbles would be placed in the left-hand side of the balance with three visible ones, and eight marbles would be placed in the other (to balance the scales, an empty bag would also need to be placed in the right-hand pan); see Figure 1.6. Although the potential exists to use this model to develop algebraic reasoning with upper primary pupils, at a basic level it provides a visual representation of the 'missing number' type questions (such as those in Figures 1.2 and 1.3) that can be explored by young children.

To begin the move towards symbolic representation, this idea could then be extended to cards on which the objects are represented pictorially (see Figure 1.7). Following on from this, pupils would be encouraged to make their own marks to record, eventually leading to the written form.

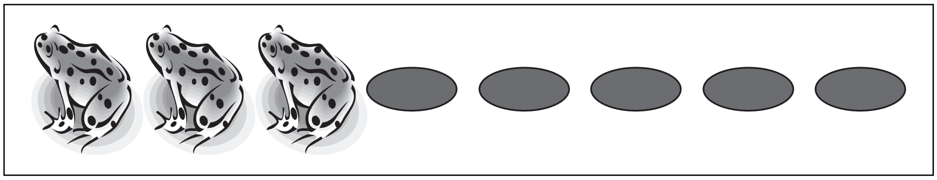
It is of fundamental importance that we consider the difficulties pupils experience with equality when selecting visual images and apparatus to illustrate operations – subtraction in particular. Instead of showing 'before and after' models with the subtracted objects removed completely in the 'after'

Figure 1.7 Picture cards used to represent objects in the equality relationship 2=3 –1



* Note the subtracted 'banana card' has been flipped over rather than removed

Figure 1.8 An example of a pictorial model that could be used a basis for discussion of number sentence structures



image, it is helpful if we leave a symbolic representation of them behind so that children can see the whole number sentence and remind them of what operation has taken place in order to achieve balance. Even modest changes to practice such as crossing out instead of erasing the 'eaten' sweets from the board, flipping over rather than removing picture cards (as in the example in Figure 1.7) or drawing empty lily pads to represent missing frogs (see in Figure 1.8) could potentially have profound influences on pupil understanding.

In Figure 1.8, a complete set of 8 can be seen (8 lily pads) as well as the partitioning into subsets of 3 (occupied) and 5 (unoccupied). With careful use of language such as 'How many more frogs would be needed so that all of the lily pads are covered?' or 'If there were 8 frogs to start with, how many have hopped away to leave 3?', images similar to the one in Figure 1.8 could be used a basis for discussing problems such as 3 + ☐ = 8 or 8 – ☐ = 3 and, in doing so, help pupils to develop their familiarity with a range of number sentence structures.

Concluding remarks

The notion of equality is a central – but sorely neglected – concept in mathematics education. There is considerable scope to introduce children – albeit often unwittingly – to strategies which have the potential to create misconceptions in their thinking and thus cause difficulties in their future mathematical development. What is more, studies on older pupils (e.g. Knuth et al., 2006; and Kieran, 1981) have identified a strong relationship between understanding of the equals sign and the ability to solve algebraic equations. Thus, perhaps primary teacher education should not confine its mathematics curriculum to a narrow age range: Kath, for example, explained that equality 'wasn't taught to me as I was trained for upper primary'. Ian also suggests that there is a 'case for at least getting more talking between year groups in school'.

Many of us are tempted to use a particular tried and tested model or phrase which we know from our own experience works very effectively for children of a particular age. Indeed, using analogies and endeavouring to simplify situations is a natural, and entirely reasonable, part of the teaching process. However, teachers need to have a greater understanding of the broader mathematical context in which they do this. Overemphasis of a particular model can make it more difficult for children to solve problems when they are presented in unfamiliar contexts. Sandra (year 2 teacher) stressed the importance of varied and accurate 'teacher talk', commenting that:

> Children have better understanding if teachers use a range of vocabulary associated with the equality symbol such as the 'same as' as well as 'total'.

This is supported by observations made by other contributing authors in this book who noted that children had a better understanding of equality after teachers had encouraged them to articulate number sentences just as they would read sentences in literacy. Ruth (also a year 2 teacher) highlighted the need for teachers to vary the position of the equality symbol and avoid always setting out number sentences in the same 'left to right' format $a + b = c$, adding

> the further up the school they go the more they will have to get use to seeing it in different ways.

Such issues are discussed in more detail in Chapter 5.

Summary of key ideas

- Children see '=' as an instruction to complete an operation. Emphasise the equivalence aspect of the '=' symbol by using phrases such as 'is the same as' or 'gives the same result as' in preference to 'makes' or 'leaves'.
- Use concrete apparatus such as balances and visual images to represent a variety of number sentence structures with the 'unknown' on both the left- and right-hand sides of the equals sign.
- Take care with how you use the '=' sign when demonstrating complex problems with multiple steps. Use arrows if it is necessary to link the successive stages together.
- Talk to colleagues teaching older and younger primary classes. What mathematical misconceptions are commonly held? Try to discover their origins and work together to develop strategies to prevent their occurrence and perpetuation.
- Children can be very innovative when presented with unconventional problems, and their responses can reveal much about both your teaching and their mathematical understanding.

Further reading

Haylock, D. and Cockburn, A. (2008) *Understanding Mathematics in the Lower Primary Years*. London: Paul Chapman.

Chapter 1 of Haylock and Cockburn explores the various ways in which children interpret the equals sign and pays particular attention to the notions of transformation and equivalence as well as offering practical activities to use in the classroom.

Anghileri, J. (2000) *Teaching Number Sense.* London: Continuum.

The third chapter of this very accessible text provides interesting insights into how children develop their understanding of mathematical symbols.

Cajori, F. (1928) *A History of Mathematical Notation: Notations in Elementary Mathematics* (Vol. 1). Chicago: Open Court.

Although 80 years have passed since it was first published, Florian Cajori's two-volume work continues to be reprinted and has been described by a reviewer on Amazon.com as

> unsurpassed … this history of mathematical notation stretching back to the Babylonians and Egyptians is one of the most comprehensive written. … Florian Cajori shows the origin, evolution, and dissemination of each symbol and the competition it faced in its rise to popularity or fall into obscurity.

This is certainly not an 'entry level' text by any means, but I would recommend this book to mathematically inclined readers who also have an interest in history.

References

Behr, M., Erlwanger, S. and Nichols, E. (1980) How children view the equal sign. *Mathematics Teaching,* 92, 13–15.

Cajori, F. (1928) *A History of Mathematical Notation: Notations in Elementary Mathematics* (Vol. 1). Chicago: Open Court.

Cockburn, A.D. (2007) Understanding subtraction through enhanced communication. In A.D. Cockburn (ed.) *Mathematical Understanding 5–11.* London: Paul Chapman Publishing.

Desforges, A., and Desforges, C. (1980) Number-based strategies of sharing in young children. *Educational Studies,* 6, 97–109.

Falkner, K.P., Levi, L. and Carpenter, T.P. (1999) Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, 6, 232–236.

Freiman, V. and Lee, L. (2004) Tracking primary students' understanding of the equality sign. In M. Hoines, and A. Fuglestad (eds), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 415–422). Bergen: Bergen University College.

Gelman, R. and Gallistel, C.R. (1978) *The Child's Understanding of Number.* Cambridge, MA: Harvard University Press.

Hughes, M. (1986) *Children and Number.* Oxford: Blackwell.

Jones, I. (2006) The equals sign and me. *Mathematics Teaching,* 194, 6–8.

Kieran, C. (1981) Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 159–181.

Knuth, E.J., Stephens, A.D., McNeil, N.M., and Alibali, M.W. (2006) Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37, 297–312.

Saenz-Ludlow, A., and Walgamuth, C. (1998) Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, 35, 153–187.