GRAPH ALGEBRA

Mathematical Modeling
With a Systems Approach

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1. SYSTEMS ANALYSIS

Rarely has a new approach to theory development offered as great a potential for impacting the way social scientists develop mathematical models of social and political phenomena as is the case with graph algebra. Graph algebra is both a tool and a language that originates from systems theory. It was originally developed by three social scientists, Fernando Cortés, Adam Przeworski, and John Sprague, in a seminal volume *(Systems Analysis for Social Scientists)* that first appeared in 1974. In this book, I both explain and extend the language of graph algebra, as well as update its application to address contemporary mathematical and social-theoretical themes. Thus, this book is not merely a reference work of ideas previously presented elsewhere; there also is a great deal of entirely new material in these pages. This reflects the fact that graph algebra—like any living language—continues to grow as the theoretical needs of social scientists expand and evolve. That graph algebra continues to speak to the needs of social and political theorists is a testament to the power of the ideas of its three originators.

In its essence, the use of graph algebra assists social scientists in developing new and surprisingly sophisticated mathematical models of complex social phenomena. This is true for both linear and nonlinear models. Social scientists use the graph algebra language to translate social scientific theories into mathematical formulas or models. Indeed, a creative thinker can often use graph algebra to algebraically "flesh out" even the most complicated and sophisticated of theories. Importantly, graph algebra can empower social scientists to "escape" from a dependence on simple linear regression models that are based on rudimentary intercepts and slopes that reveal little more than correlations within a set of variables. Moreover, regression can often be creatively applied to fully estimate intellectually appealing graph algebra models using commonly available statistical software. This allows social scientists to incorporate greater theoretical depth in the algebra of their models while still utilizing known statistical procedures.

Some readers will find this book to be highly correspondent with recent initiatives in theory building in the social sciences, perhaps best typified by efforts pursued by the National Science Foundation. Not long ago, the National Science Foundation launched an initiative called "The Empirical Implications of Theoretical Models" (EITM). The EITM (2002) report from the National Science Foundation states,

> A schism has developed between those who engage in formal modeling that is highly mathematical, and those who employ empirical modeling which emphasizes applied statistics. As a consequence, a good deal of research in political science is competent in one technical area, but lacking in another, that is, a formal approach with substandard (or no) empirical tests or an empirical approach without formal clarity. Such impaired competency contributes to a failure to identify the proximate causes explicated in a theory and, in turn, increases the difficulty of achieving a meaningful increase in scientific knowledge. (p. 1)

While this book is aimed at all the social sciences (and thus is by no means limited to the discipline of political science), the observation that efforts are needed in many areas to better link social and political theories to testable empirical models is worth noting. The question is, "How does one do this?" And even more pointedly, "How does one teach someone to do this?" Coming up with an intellectually interesting algebraic specification has historically been one of the most challenging things most researchers have ever had to do, which is why so many scientists rely on the "canned" linear regression model. Some scholars, after noting the difficultly and seriousness of the problem, have argued that researchers can either search for their own model specification based on their own theory or, failing that, perhaps use statistical methods that impose no structure on the analysis. This latter approach would use "theory-less methods" (see, e.g., Signorino & Yilmaz, 2003). What this book offers is a means by which such linkages can be made using a highly practical new graphical language that empowers social scientists to develop nuanced algebraic models of their theories that contain a level of intellectual sophistication that might have previously appeared forbidding, or perhaps even impossible.

There are a number of highly prominent examples of the use of graph algebra in the social sciences. Sometimes graph algebra is used in the theory-building stage of a project to assist in the development of a model's algebra for a social or political process, but the graph algebra itself is not presented in the final printed report. This is the case with the seminal volume *Paper Stones: A History of Electoral Socialism* by Przeworski and

Sprague (1986), which contains a theoretically rich analysis of the development of leftist voting in Europe during much of the 20th century. In this case, while graph algebra was used early in the research to develop a sophisticated model of leftist voting, the book itself presents (and explains) only the normal algebraic version of the model.

In other cases, the graph algebra itself is presented in the final report of a research project as a means of helping to simplify the presentation of a complex model that might otherwise appear inhibiting for some readers. This indeed is the case with research by Duvall and Freeman (1983), in which they develop and analyze a model that helps explain how certain elites dominate the industrialization processes in many developing nations. This is also the case in some of my own research in which I present a model of congressional voting in the United States using graph algebra to help readers retain a wide-angle view of a somewhat complicated political theory (Brown, 1991, see especially the appendix to Chapter 7). In other cases, graph algebra has been effectively included in a final report as a means of emphasizing the linkage between the model and the social or political theory. This is often useful even in situations in which the normal algebra itself is not so intimidating, as in the case of an analysis of electoral institutionalization and voter mobilization in many European nations by Przeworski (1975). Thus, graph algebra can be used "behind the scenes" to develop a sophisticated algebraic specification of a complex theory, or it can be used in a more up-front manner that also assists with the presentation of the theory itself. Either way, researchers can use graph algebra to help develop, analyze, and present social and political theories that incorporate surprising levels of intellectual richness.

Before describing the details of graph algebra, it is worth placing it in the context of systems theory. Why approach mathematical modeling from the perspective of systems analysis? Many scientists—whether they work in the social sciences, physical sciences, natural sciences, or engineering—often think in terms of systems. Just about everything influences something else, which in turn either feeds back into itself or affects something different. Consider human organizations. We live in systems. Our nations, towns, international associations, friendship networks, and families are systems. We have court systems, electoral systems, presidential systems, parliamentary systems, and bureaucratic systems. Our small groups and associations are systems, just as an inner-city gang is a system regulated by norms that are enforced with punishments and rewards that maintain membership, identity, and coherence. The list of systems that surround us goes on and on, seemingly without end.

Our biology is organized in terms of systems. Our bodies are living systems, as are each of the smaller components of our bodies, such as our cells, our nervous systems, our immune systems, our digestive systems, and our reproductive systems. Each system depends on another system for its functioning and/or survival, with each level adding new complexity to the macro-organization of higher levels of systems. The ecology of a lake is a system of species, with each species having a direct or indirect influence on the others. A wolf den is a system with a clear sense of hierarchical dominance. An ant colony is a system with a highly differentiated work hierarchy.

Our physical environment is organized in terms of systems. Houses are filled with systems. The regulation and control of water in our toilets functions as a system. We live in temperature-regulated environments, with heating and air conditioning apparatuses that are controlled as a system. Even a toaster oven is a self-regulating system.

The smallest and largest parts of our universe are organized as systems. Atoms are systems that we identify by name in the periodic table. Entangled photons are quantum systems. Galaxies are systems, as are nebulae that give birth to new stars. Our own solar system is a system that both inspires us and helps us to measure time and track the seasons.

We organize our thoughts systematically, in the sense that we arrange different thoughts such that they complete a pattern that is itself an identifiable conceptual entity. Indeed, the way in which we process our thoughts is a system that responds to inputs (stimuli, new information, etc.) and produces outputs (physical movement, decisions, etc.). It is natural for us to think in terms of systems because we are enmeshed in them at every level of our existence. For this reason, it is inevitable that we would seek to use a language that helps us describe our systems in a manner that lends itself well to analysis. This need gives rise to "systems analysis."

The term *systems theory* means different things to different people. Early pioneers in the field now known as *general systems theory* include Ludwig von Bertalanffy (1976), W. Ross Ashby (1956), and Gerald M. Weinberg (1975). By most estimates, the large and diverse general systems literature dates back to the 1940s, although the precursors to general systems theory predate even that. James Grier Miller (1978) made a major contribution in expanding the definition and scope of *general living systems theory,* a more specialized approach to general systems theory as it applies to living organisms of all types. On the other hand, electrical engineers look at systems theory from the perspective of the flow of electrons through circuitry. Other engineers see systems theory from the perspective of mechanical inputs and outputs. Computer programmers look at systems theory from the perspective of code that specifies a sequential set of instructions. Still others think of systems theory in other ways.

The focus of this book is on a specific set of mathematical tools that arise mostly from systems theory as it is encountered in both electrical and physical engineering. But we address social systems here, not engineering systems, regardless of where the mathematical techniques came from originally. Also, no effort is spent trying to systematically integrate the material presented here with the general systems literature. Indeed, there is only a tangential link between the contents of this book and the way systems theory is applied in fields outside of the social sciences. The only substantial connection of these other approaches to systems theory is the consistent theme that one can investigate a system by understanding how its internal parts are arranged and how they operate in a coordinated manner.

Systems theory as described in this book dates back to the seminal book by Cortés et al. (1974). In that volume, the authors translated and reinterpreted mathematical methods that are predominantly described in the engineering literature such that these methods could be productively used by social scientists. While their efforts were successful from a utilitarian point of view, there are nonetheless considerable superficial differences in the way social systems are described using their approach when compared with styles common to the engineering disciplines. Most notably, Cortés et al. describe systems using "graph algebra," which engineers will more commonly reference as "block diagrams." Also, engineers tend to favor "signal-flow graphs" rather than block diagrams since they are more compact. However, the compactness of signal-flow graphs leaves them both more terse and abstract than would be helpful to most social scientists. Outside of the engineering literature, block diagrams are also quite commonly used to describe more general and nonmathematical process-related ideas connected to general systems theory, and this intellectual baggage would only serve to confuse a new application of systems mathematics as it applies to the social sciences. Thus, Cortés et al. chose to mix the block-diagram approach with the algebraic utility of signal-flow graphs, which led to the term *graph algebra* to identify a mathematical style of representing systems that was uniquely tailored to describing social scientific phenomena.

Graph algebra is best described as a language for translating social scientific theories into mathematical formulas. The language is designed to ease the model-building process such that users of graph algebra can develop more sophisticated models of complex social scientific ideas than might otherwise be possible. We currently do not know the limits of how broadly graph algebra can be applied to social scientific questions as it is still quite commonplace for researchers to propose and develop new uses and innovative applications of the language. For example, there have been significant advances in the mathematics of dynamical systems since graph algebra was first invented. This is particularly true with respect to systems of differential and difference equations, chaos theory, and catastrophe theory. Thus, this book both describes and extends the language of graph algebra, as well as updates its use with respect to some of these new applications.

Structure and Function

At the core of systems theory is the identification and description of a system's structure, its function, and the response of the system to inputs. In the most basic terms, a system's structure is the collection and arrangement of its parts. The system's structure is essentially unchanging. Some social scientists may object to the notion that anything in human affairs is static, and I have no fundamental quarrel with this position. However, there are many things that remain approximately the same for a long period of time, and for our purposes this is sufficient for us to proceed with the description of social systems. For example, it is true that all democracies evolve, and with that evolution comes change. But the electoral systems of most democracies persist unchanged in their essential characteristics for years, and often decades. Change can occur when, say, in the United States the voting franchise was extended to women in 1920, when African Americans were able to vote in large numbers with the banishment of the Jim Crow laws in the 1960s, or when the voting franchise was extended to 18- to 20-year-olds in the 1970s. But between those years, and again after the 1970s, the electoral system in the United States has changed very little in terms of its structure. Moreover, if we reconceptualize vote mobilization as an input into an electoral system rather than as a part of the structure of the system, we can argue that only the mobilization inputs have changed and that the voting system itself has changed very little since the American Civil War.

We do not directly observe the structure of a system in the manner in which we observe, say, an apple. The structure is only a theoretical concept. It says that certain rules are followed that organize human activity. For example, the division of the American electoral system into separate states and congressional districts is part of the structure of the system. The fact that a concomitant presidential election every 4 years boosts voting turnout in every other congressional election (which occurs in a 2-year cycle) is a feature of the structure of the American electoral system. The use of a winner-take-all ballot and the consequent promotion of a two-party system is part of the structure of the system. In a parliamentary system of government, proportional representation is part of the structure of the system. In governments that have hybrid forms of democratic representation, presidents, prime ministers, district-based parliamentary seats, and party lists are all parts of the structure of an electoral system.

Consider some of the many rules that surround us as members of any society. These rules are elements of larger structures that we can identify as the infrastructure that both creates and supports our social systems. For example, in most societies there are rules against marrying close relatives.

Stock market trading prohibits insider disclosures of company information. The Federal Reserve Board in the United States meets in private and often delays the reporting of its deliberations to avoid volatile market reactions. The trading of endangered species is prohibited. Gases that destroy the ozone layer in our atmosphere are often restricted in their use. The safety of the workplace is often regulated in developed countries. Zoning restricts the use of private property. Price controls place restrictions on the natural interplay between demand and supply. All these examples are parts of structures for complex systems within which we live. We do not physically see these structures, but we know the rules that define them. When we talk about "social theory," we are really discussing the identification and analysis of social structures. We investigate social structures to understand why they produce the social reality that we actually do observe.

All system structures provide a function. The function of a system is what the structure does. In elementary systems, the function of a system transforms the inputs into outputs via a forward path. That is, a system's structure receives an input, and that input is changed in some way to produce an output. In slightly more advanced systems, there can be a feedback path in which the output reenters the system as a new input. This is how feedback works in the case of a microphone that picks up sound from speakers and then feeds that sound back into the amplifier, which then sends it back out to the speakers, and on it goes until a loud squeal is produced by the speakers. In that case, the structure of the system is the microphone, the amplifier, and the speakers, all of which are connected in a certain order. The function of the system is the transforming of input sounds registered by the microphone into output sounds produced by the speakers. In the absence of feedback, the amplifier is the only element of the forward path that connects the microphone to the speakers.

Thus, a system's structure transforms inputs into outputs via its function. The inputs vary, and the outputs vary. Intermediate states of the system that exist after the inputs have gone into the system but before the outputs are in their final form also vary. But the structure of the system is fixed, or invariant. This means that the rules that define the structure of the system remain static for a sufficiently long period of time to allow us to investigate the system's functioning.

Thus, when we speak of systems, we are normally examining synchronic change. This is change as found in the inputs and outputs, but within the context of a system's structure that is invariant. When the system's structure changes, then we have diachronic change. Diachronic change normally means that a new system has taken over and that the old system is obsolete. In practical terms, diachronic change typically requires social scientists to develop a new theory, and a new model. However, it is worth pointing out at this juncture that systems can sometimes be sufficiently sophisticated in their structure such that what once may have been considered an emergence of diachronic change can be explained within the context of a single model's experience of synchronic change. For example, this is a prominent feature of models that employ catastrophe theory (see Brown, 1995b).

An Overview of the Substantive Examples Found in Subsequent Chapters

The greatest strength of graph algebra is its flexibility—as a language—in working within a great variety of substantive contexts. For this reason, I have attempted to show throughout this book how graph algebra works using a diverse collection of examples. Some of these examples are very basic, as would be needed heuristically to convey the essential mechanics of the graph algebra grammar. But other examples are much more extensive, raising new and potentially provocative specification questions regarding well-known models that have existed in the extant social science literatures for some time.

In Chapter 2, I begin by using graph algebra to ‘‘re-create’’ the linear regression model. This is an especially useful example because it allows the later (and more complex) models to be placed in stark contrast with the nearly overbearing simplicity of the linear model. I then develop a simple voter mobilization model that applies some of the basic feedback capabilities of graph algebra and finish the chapter by using graph algebra to develop the well-known Keynesian multiplier from economics.

Chapter 3 further extends the voter mobilization model as I introduce the use of time operators with graph algebra, especially in systems involving feedback and control. In one case, I discuss this with respect to the elections held in Iraq in 2005. But I also discuss how these ideas can be related to other substantive areas, such as population growth in China. At the end of Chapter 3, I show one approach to estimating graph algebra models using an example of labor union membership in the United States.

Chapters 4 and 5 describe how graph algebra can be used with systems of equations. In both these chapters, I develop an extensive and running example of the arms race model of Lewis Fry Richardson. I first show how Richardson's original model can be specified using graph algebra and then I demonstrate how his model may in fact be a reduced-form version of a number of more complicated models. The more complicated models often seem to capture the substantive intent of Richardson's ideas more than his original specification, and these examples are used to demonstrate how easy it is to work theoretically with graph algebra. I also demonstrate one approach to estimating these models using ordinary least squares. While Chapter 4 focuses on the setting of discrete time, Chapter 5 moves the discussion to continuous time models.

Chapter 6 introduces the rich topic (and one of my personal favorites) of how to use graph algebra to work with nonlinear systems. While there are a great many algebraic approaches to nonlinearity, I organize the discussion beginning with relatively simple mechanics involving nonlinear filters before turning to more complex forms. As the chapter progresses, I explain how graph algebra can be used to model various versions of logistic growth, a common feature of many population models. I also address how graph algebra can be applied relatedly to the ideas of concurrent and lagged environmental decay, global warming, and rising sea levels. The chapter then turns to the issue of chaos, and graph algebra is used to specify the equations that produce Lorenz's well-known chaotic strange attractor. Finally, I return to Richardson's arms race model by showing how one can explore this model in combination with a forced oscillator.

Chapter 7 introduces the idea of using conditional paths with graph algebra. I begin by describing how conditional paths relate to important theory-building ideas that are current in the sociological literatures. I then show how graph algebra can work with decision or choice theory. It is possible to use choice theory to manipulate the structure of a system as it is operating in real time. Applications include a revised look at Richardson's arms race model, as well as how to model the transition from authoritarian to democratic rule in China using catastrophe theory. These ideas allow theorists to blend the two worlds of individual choice and stimulus-response that so often (and perhaps unnecessarily) seem at polar opposites of the theoretical spectrum.

In Chapter 8, I show how shocks and other forms of stochasticity can be introduced into a graph algebraic representation of a system. Here, I return to the logistic model and offer an example of how this model may be modified using graph algebra to address the issue of population growth in the context of an environmental disaster. I then extend this discussion to a similar context, but with respect to a rapid rise in oil prices. Both these issues are highly topical in contemporary times, and researchers will want to use graph algebra to add realistic complexity to their models of these and other vitally important topics.

Chapter 9 turns to the issue of the implied theoretical content of the graph algebra language itself. Here, I categorize three types of system equilibria, system stability, variable stability, and meta-equilibrium, within a system cascade. Here is also where I approach the subject of graph algebra using the broadest theoretical brush, and I relate the issue of system stability to ideas of societal development raised by theorists such as Nisbet, Rostow, Organski, Ingelhart, and Pye. Graph algebra need not be limited to those realms in which one wants to resolve a parameter's value. Indeed, graph algebra can be used narrowly to specify a model for a specific question or problem or broadly to theorize within intellectual realms of significant expanse, or anywhere in between.

2. GRAPH ALGEBRA BASICS

The use of graph algebra can yield marked benefits to theory building in the social sciences, and it is useful to view these benefits when considering the linear regression model. Arguably, the most common model used in the social sciences is the linear regression model. While many approaches to parameter estimation exist for linear models, the ultimate result is typically a table with a list of independent variables and their associated parameter estimates and standard errors. From this perspective, the list of variables in the table *is* the model. Specification concerns usually revolve around the question of whether or not a researcher has omitted one or more important variables from the analysis, although sometimes the issue of functional form also is involved.

While graph algebra does not reduce a researcher's need to be aware of potential omitted variable specification problems, it does allow the researcher much greater flexibility with respect to designing innovative and intellectually appealing functional forms. As an absolute minimum, graph algebra allows us to develop more sophisticated model specifications such that the algebraic form of the model becomes as important as the variables that exist within that form. Thus, systems theory as it is expressed through graph algebra offers a means of developing algebraic formulations that correspond with social and political theories that are more complex and sophisticated than the ubiquitous linear form. Thus, as a movement away from the linear model, the use of graph algebra encourages the development of increasingly interesting scientific theories. Moreover, as will become clear by the end of this book, such theories find their origin in the thinking of the theorist, not in the graph algebra itself.

A researcher gains the benefits of graph algebra by mastering its functionality as a language. Graph algebra is the language that we use to describe a system's structure and functioning. With graph algebra we identify the parts of the system's structure, and then we connect those parts in a process that identifies the structure's functioning. Thus, the system's