What Your Colleagues Are Saying . . .

Whose Math Is It? is a must-read for any math teacher who seeks to create a classroom learning environment in which their students are actively engaged in challenging mathematics, thoughtfully and respectfully discussing their mathematical thinking, and personally aware of their own learning and agency. Whether you are new to or experienced with active learning, this book offers a wealth of concrete strategies that will expand and enrich your instructional repertoire. A book like this with such experience-based insights is a treasure and does not come along very often. I highly recommend it!

—**Chris Rasmussen**, PhD in mathematics education

Whose Math Is It? is about teaching students how to take ownership of "their math." A comprehensive book that includes examples, tools, strategies, and resources, it is great for any educator (teacher, co-teacher, instructional coach, or administrator) who wants all their students to see themselves as mathematicians who critically think about mathematics and talk about it in meaningful ways. Educators who want their math students to feel confident and believe in themselves will want to read this book!

—**Staci Benak**, EdD, math resource teacher, San Diego Unified School District

Student efficacy in the math class is attainable and should be a goal for every math teacher. *Whose Math Is It?* provides effective strategies to move the focus from teachers doing the heavy lifting to students becoming empowered in their learning. Joseph Michael Assof's book guides teachers in the creation of classroom systems that support student agency in learning.

—**Kim West**, Corwin faculty member, Kramer IB World School
PYP coordinator, and math instructional coach, Dallas ISD

Whose Math Is It?

Whose Math Is It?

Building Student Ownership in Mathematics

Joseph Michael Assof

Foreword by Douglas Fisher

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Foreword

By Douglas Fisher

I have had the opportunity to observe Joseph Assof teaching on hundreds of occasions. He is an expert, skilled at guiding students' mathematical understanding, building their confidence and competence. Students leave his classes with more than procedural knowledge. They gain conceptual understanding and can apply what they have learned to novel situations. In other words, they reach the level of transfer, generalization, application, and authentic use of their knowledge.

Part of Joseph's belief about learning, which he has shared with teachers across the country, is that students have to develop ownership of, and responsibility for, their mathematics learning. This book is aptly titled *Whose Math Is It?* because Joseph knows that students must develop efficacy in their mathematical learning if they are to take responsibility for their learning. Students must have goals and align their efforts along with their goals. And they must experience the fruits of their labors, knowing that they are learning more and better as a result of their efforts.

This is no small feat. Far too many students approach mathematics with the belief that they will fail, that math is for other people, and that they are not capable of learning the content. We all recognize this fallacy, but Joseph shows us how to teach students that this is not reality; that they own their mathematical learning.

To accomplish this, Joseph embraces teacher clarity. In fact, he is one of the original authors of the *Teacher Clarity Playbook*, a book that outlines a process for analyzing standards and designing learning experiences. But teacher clarity is more than learning intentions and success criteria. It's also the meaningful experiences students have with the content.

Clarity requires that educators match instructional approaches with the appropriate phase of learning. Importantly, these phases can occur within a single lesson or across multiple lessons. Surface learning is not superficial, it's foundational or introductory. And there are tools teachers use to build students' surface learning. But we can't leave students there. When teachers change instructional approaches and tasks, they can move students to deep learning, during which time students make connections, see relationships, and develop schemata. Our goal is not adult-dependent learners but rather students who self-regulate and continue learning. At the transfer level, students can apply their learning in new situations. As I have noted before, it's the right approach, at the right time, for the right kind of learning.

Transfer of learning, the goal of our collective efforts, is not easy. In fact, the American Psychological Association (2015) notes that "student transfer or generalization of their knowledge and skills is not spontaneous or automatic" (p. 10). And that's the magic of this book. By teaching students that the math is theirs, that they own it and use it, students begin to transfer their learning. Joseph shows that there are processes and procedures that teachers can use to guide students' thinking without telling them what to think. Filled with examples across the grade levels, this book supports educators in developing a mindset with students that they are the owners of mathematics, that they can use math to solve interesting problems, and that they are responsible for their own learning.

About the Author

Joseph Michael Assof is a high school and community college mathematics teacher and the math department chair at Health Sciences High and Middle College in San Diego, CA. He is also an educational consultant and presents internationally on a wide array of topics including teacher clarity, mathematics teaching and learning, visible learning, and more. Joseph coauthored *Teaching Mathematics in the Visible Learning Classroom, High School*, *Teaching Mathematics in the Visible Learning Classroom, Grades 6–8*, and *The Teacher Clarity Playbook*, and his classroom is featured in a number of *Visible Learning for Mathematics, Grades K–12* videos. Joseph holds bachelor's and master's degrees in mathematics and a doctorate in Educational Leadership with an emphasis in Mathematics Teacher Leadership. Mathematics and mathematics education are Joseph's second passion—his first being his two beautiful boys, Joseph Fred and Jamie Beau.

Introduction

Keeping the End in Mind

Imagine a mathematics classroom where students are not only actively engaged in critical thinking, problem solving, and constructive argumentation, but where they are also aware of their own learning, seek feedback on their work, provide feedback to their peers, and monitor their own progress. One can imagine further that there exists a culture in this classroom of high expectations sustainable only by its equally prominent culture of support.

This is a classroom fueled by efficacy—where students are choosers and users of learning strategies that have proven effective for them in the past and thus give them confidence to use them again. This is a classroom where the teacher may truly embody the role of facilitating learning, with confidence that their expertise is not going underutilized. Now, compare this abstract ideal to the concrete reality. This comparison might tempt some down the student-by-student road; checking off individual talented students who could rise to the occasion of such an idealized classroom and crossing off others who likely would not. This approach, however, begs the question: Do we develop or select talent? And while many of us in education might instinctually and fervently (and commendably) react to such a question, without efficacy of our own, the prospect of developing such a high degree of talent might seem unattainable.

Thus is the purpose of this text. This book seeks to act as the representational intermediary between the abstract ideal classroom described above and the concrete realities of our own classrooms. This text is designed to help mathematics teachers realize the ideal and bring the abstract to the concrete through key practices targeting the development of student ownership of learning. For when asked the question *Whose math is it?* every student should respond, *My math!*

The Role of the Students

Think about the students in your classroom. How do they see themselves as participants in the mathematics classroom community? Further, how do they see themselves in respect to math itself? Some students consider themselves to be passive recipients in the mathematics classroom—why is this? Math, to them, is likely a large collection of facts and procedures that need to be unveiled by an expert so they can be apprenticed into recall and reproduction. In this sense, mathematics is much like tradition in that it must be passed on to survive—if all the math teachers suddenly vanished we would never know math again! (Something that would likely land with minimal tragic impact to the students described here.) These students don't have a *say* in mathematics—no one does! Mathematics just simply *is*.

Contrast the mindsets of these students with those in the classroom previously described, where students are clearly positioned as problem-solvers with agency over their learning. They have a stake in the game, they lean into challenge, and they believe progress will come with effort. To those with agency in the subject, mathematics is something that can be—and *needs to be*—discovered individually and collectively. The ability and authority to validate mathematical claims, check the accuracy of calculations, and determine the reasonableness of solutions lives within them—not beyond them. They may appreciate external validation, but it is not prerequisite to confident progress. To these individuals, math is *personal*, math is *owned*. These individuals cannot be told that $1 + 1 = 47$, for they have independent access to the existential structure of mathematics where this falsehood doesn't pass the smell test. Simply put, these individuals are mathematicians.

Surely we have had students arrive in our classrooms with mindsets on both ends of the spectrum outlined here—as well as in many places in between. The question for us as teachers becomes, how do we take students from wherever they are and help them develop more of the ownership required to be successful in mathematics? In order to do this, however, we need a benchmark understanding of their foundational starting point. One way to do this is by using the *Student Mathematical Ownership Itinerary* (Table I.1 and Table I.2). This tool can be used to inform you (and your students) how each learner situates themselves in the mathematics classroom and in respect to math itself. It can be used at the beginning of the school year as a pre-assessment of mathematical agency, as a formative benchmark throughout the school year to inform your instructional decision making, and at the end of the year to measure the impact of your approach.

Table I.1 Student Mathematical Ownership Itinerary

STUDENT MATHEMATICAL OWNERSHIP ITINERARY			
State the degree to which you agree with each statement below.			
1. I can use math as a tool to make sense of the world.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
2. Math is a large collection of facts and procedures that need to be memorized.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
3. I can discover math on my own.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
4. I need a teacher to show me how to do math before I can learn it.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
5. I can make choices when doing math about how I want to solve a problem.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
6. There is one right way to do math.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
7. I can check my own work to see if I did it right.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree
8. I need a teacher to tell me if my answers are right.			
[ ] Strongly Agree	[ ] Agree	[ ] Disagree	[ ] Strongly Disagree

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Table I.2 Student Mathematical Ownership Itinerary (version 2)

STUDENT MATHEMATICAL OWNERSHIP ITINERARY			
Read each statement. Circle the picture that matches how you feel.			
1. I can use math as a tool to make sense of things around me.			
Strongly Agree	Agree	Disagree	Strongly Disagree
2. Math is a group of facts and steps to take that I need to memorize.			
Strongly Agree	Agree	Disagree	Strongly Disagree
3. I can figure out math on my own.			
Strongly Agree	Agree	Disagree	Strongly Disagree
4. I need a teacher to show me how to do math before I can learn it.			
Strongly Agree	Agree	Disagree	Strongly Disagree
5. I can make choices when doing math about how I want to solve a problem.			
Strongly Agree	Agree	Disagree	Strongly Disagree
6. There is only one right way to do math.			
Strongly Agree	Agree	Disagree	Strongly Disagree

STUDENT MATHEMATICAL OWNERSHIP ITINERARY			
7. I can check my own work to see if I did it right.			
Strongly Agree	Agree	Disagree	Strongly Disagree
8. I need a teacher to tell me if my answers are right.			
Strongly Agree	Agree	Disagree	Strongly Disagree

Source: Smiley icons courtesy of iStock.com/Makrushka

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To score this assessment, assign a scoring scale of 3: Strongly Agree, 2: Agree, 1: Disagree, and 0: Strongly Disagree to all odd numbered statements and a reversed scale of 0: Strongly Agree, 1: Agree, 2: Disagree, and 3: Strongly Disagree to all even numbered statements. Scores of 0–10 indicate low perceived ownership of mathematics, 11–16 indicate a moderate ownership of mathematics, 17–24 indicates a high level of student ownership of mathematics.

To be clear, I am not trying to send the message that students arrive in some sort of a fixed manner regarding mathematical ownership whereby some have it and some simply do not. Rather, this initial focus on the role of the student is meant to highlight the impact of their surroundings and learning environments—including their teacher—on their presumed capacity for mathematical ownership. In other words, as teachers, we have great influence over how students position themselves with mathematics. The language we use, the environments we foster, the tasks we launch, the ways we interact with others—all of this impacts how students are positioned in the content and our classroom/math course. That's great news! It means we have the power to affect positive change in our students' sense of self. If, that is, we act with intention. In the next section, I will seek to further illustrate how our decisions and actions as teachers produce much more than just marks on papers.

The Role of the Teacher

Think about our primary role as teachers of mathematics. Are we disciples of the subject, facilitators of learning, or perhaps, both? Consider the following exchange between a student and teacher during a middle school lesson on using variables to represent quantities in a real-world problem. Students are independently working on the following problem while the teacher circulates the room.

The perimeter of a rectangular swimming pool is 54 meters. The length of the pool is 6 meters. What is its width?

Student: *[raises their hand and signals the teacher over] Is this right?*

Teacher: *Can you tell me what you did?*

Student: *OK. Well, I wrote* $6 \cdot w = 54$ *because the formula is* $l \cdot w$ *and then just divided 54 by 6 and got 9 for w.*

Teacher: *So, that's the formula for area . . .*

Student: *Ohhh . . .*

Teacher: *. . . and you want perimeter instead, which is* $2l + 2w = 54$*. So since you know the length is 6, you can write [signals to student to start writing as he speaks]* $2(6) + 2w = 54$*. Right. Now what is* $2 \cdot 6$*?*

Student: *12?*

Teacher: *Right. And now we need to subtract the 12 from both sides of the equation [points to paper to indicate the student should write what he is suggesting]. And* $54 - 12$ *is . . . ?*

Student: *42?*

Teacher: *OK so if* $2w = 42$*, then how much is just one w?*

Student: *21?*

Teacher: *That's right! Make sure you write that all down. [Continues circulating room]*

What do we notice about how the teacher and student respond to one another? The student—for one reason or another—was looking for some sort of validation of their work. Work, it is worth mentioning, that was absolutely mathematically correct, albeit misplaced on this particular task. The teacher follows the student's inquiry with an open request for explanation, which could communicate the importance of process in the class. Once the student unveils their thinking, however, the teacher assumes a corrective stance and begins walking the student through the problem-solving process. The student seems to recognize their error in problem setup after the teacher informs them that "that's the formula for area," but is quickly cut off as the teacher proceeds to plow the *correct* solution path.

Let's think about what we can infer about their presumed roles and positions within that classroom. It is difficult to discern exactly how the student might presume their own role in the classroom based on this exchange, because frankly, we don't hear much from them. The teacher, however, appears to have assumed the role of Corrector-in-Chief. Which is an important and fitting role if our primary task as math teachers is to help students produce correct answers. It is clear that the teacher has situated himself as the arbiter of truth in this exchange—the master codex against which other participants might calibrate their own efforts. Now, we should be careful here not to completely demonize the familiar "sage on the stage" metaphor—for content expertise is an invaluable tool to facilitate the many roles teachers must navigate to promote a student-centered classroom. However, the consideration I am promoting here is regarding the impact the teacher is having on the student's sense of ownership in the content and classroom/course. Namely, how is the teacher's own positioning as the *knower* and *shower* affecting that of the student? Well, we can only infer based on what we see. The student was situated to only follow instructions and answer tightly close-ended calculations. Here are some reasonable conclusions from this exchange:

- *The teacher sets up the problem, and I solve it.*
- *I need to do this like the teacher.*
- *Calculations are the important part.*
- *The way I did it was wrong.*

Regardless, are these the messages that foster student ownership in mathematics? How might this student respond if we asked them *whose math is it?*

There was a clear decision-point for the teacher in this exchange after the student explained their thinking. Let's take a look at the same exchange again, this time highlighting the decision-point, along with some additional considerations on the part of the teacher and alternative responses. We will use the expert noticing framework (Jacobs et al., 2010) whereby we first attend to the details of the case, then interpret their meaning, and finally choose how to respond.

The perimeter of a rectangular swimming pool is 54 meters. The length of the pool is 6 meters. What is its width?

Student: *[raises their hand and signals the teacher over] Is this right?*

Teacher: *Can you tell me what you did?*

Student: *OK. Well, I wrote* $6 \cdot w = 54$ *because the formula is* $l \cdot w$ *and then just divided 54 by 6 and got 9 for w.*

Decision-Point

Expert Noticing: This student is correctly using variables to represent unknown quantities and is correctly solving for those quantities. However, this student set up the problem as if they were given the area of the pool of 54 square meters rather than the perimeter of 54 (linear) meters. There is a possibility that there is confusion around units (meters versus square meters), but it could have just been an oversight, and that also isn't the primary focus of this task. There is also a possibility that the student does not know the difference between area and perimeter, but that is not clear yet, so I will need to gauge more about this. Also, I want to be careful to honor the work the student has done and situate it as legitimate mathematics, though different than what the task is seeking. So, I want to use language that validates *their* process.

Teacher: *OK. I see what you did here, and I appreciate how you used variables to represent the unknown quantities. I heard how you talked through your problem-solving process and calculations, and it all sounded mathematically legitimate to me. So here's my question . . . How would you do this if the AREA of the pool was 54 square meters instead?*

Student: *[Silent for a moment while looking at their work, and the original problem.] The area is 54? Oh, OHH!!!*

Teacher: *Yup, there it is.*

Student: *Ahhh I did area instead of perimeter! [Starts erasing]*

Teacher: *Yeah you did and . . . [waves hands] No, no! Don't erase it! That's really great work for a different problem. Maybe we should even give it to the class next? Just write the new work for this problem underneath.*

Now what do we notice about how the teacher and student respond to one another? And what can we infer about their presumed roles and positions within that classroom? In contrast to the first exchange, this time the teacher led with validation and recognition of the student's legitimate mathematical thinking—which was not contrived. Then, we saw the teacher guide the student's thinking with a targeted question that held multifaceted value. Asking the student about area provided the teacher insight into whether the student recognized the difference between area and perimeter (one of the early content wonderings), as well as served as a prompt to trigger the student's thinking around the actual *ask* of the task. The teacher did not jump into premature reteaching—which would have served as a rigor-reducing overscaffold in this case.

Further, the teacher communicated confidence in the student's own recognition of what adjustments needed to be made, which could reinforce the student's sense of ownership and efficacy in mathematics. Finally, the teacher made very clear that the student should not *undo* their original work by erasing it. This final validating move of the student's thinking could only continue to perpetuate the message that their contributions matter and their mathematical thinking is worthy. So then, perhaps some reasonable conclusions from this second exchange might include the following:

- *The teacher is here to guide me but not do the work for me.*
- *Sometimes I need the teacher, and sometimes I don't.*
- *Calculations are important but so is correctly setting up a problem.*
- *The way I did it was right but for a different problem.*

Regardless, these contrasting messages could serve to foster greater student ownership in mathematics. How might this student now respond if we asked, *whose math is it?*

Our decisions in the classroom, our choices during planning, and the way we respond to students all have the propensity to greatly affect how students see themselves as mathematicians. We have the power to contribute to or detract from our students' sense of agency and mathematical ownership—all of which contributes to their ever-dynamic identities. We need to act with care, and we need to act with intention if we are to use our powers for good. Thus is the intent of this book. How can we structure our courses, classrooms, and ourselves toward this end of promoting mathematical ownership in our students?

How to Use This Book

This book is rooted in teacher clarity and split into two parts, both presented through the context of mathematics education: determining success criteria and operationalizing success criteria. The first part, Determining Success Criteria, is intended to help teachers clearly define success in mathematics in a way that is productive for their students. We will also look at relevant research and best practices, which is the focus of Chapter 1. The second part, Operationalizing Success Criteria, is intended to help teachers provide opportunities for students to build their success and ownership in mathematics in whole-class, peer-to-peer, and individual settings through the development of social and sociomathematical norms, collaborative learning experiences, and self-regulated learning.

Sociomathematical norms: norms that are specific to a *mathematics* learning community and regulate the community's communication about and participation with the subject of *mathematics.*

Chapter 2 will explore the teacher's role in developing classwide social and sociomathematical norms that underpin the mathematical culture of their classrooms. It will also discuss how to leverage the clarity gained in Chapter 1 to explicitly develop, maintain, and leverage social norms with social learning intentions. We will see that sociomathematical norms develop in any learning community whether we intend them to or not, for better or worse—so we ought to consider shaping them with intention. Chapter 2 will further illustrate how to communicate and model the existence of *choice* in mathematics, as well as how to use discursive positioning moves to situate our students as problem-solvers with agency. The mantra for mathematical ownership at the whole-class level in this chapter is *everybody's doing it*.

Chapter 3 will discuss how to reinforce student ownership by structuring peer interactions and collaboration and will make the case for investing in collaboration as a space for students to begin taking ownership of their learning. Importantly, this chapter will recognize that students need to be primed in order to ensure that group work is indeed productive. Everything from grouping strategies to setting up and launching tasks will be covered to this end. This chapter also serves as a hub for various collaborative strategies and protocols suitable for the mathematics classroom. The mantra for mathematical ownership among students at the peer-to-peer level in this chapter is *we're doing it*.

Chapter 4 homes in on supporting individual students by promoting metacognition and self-regulated learning—essential components of ownership. It will delineate the self-directive process of self-regulation into its individual components and discuss how to scaffold students toward increased motivation by targeting each for development. This includes teaching students how to become more independent learners and study. Finally, it will demonstrate the importance of feedback and student self-assessment in self-regulated learning. The mantra for mathematical ownership for students individually in this chapter is *I'm doing it.*

The book closes with a review of the student-facing mantras of this book and their implications, as well as provides some teacher-facing mantras to guide classroom policies and decision making. Implementation is as much about mindset as it is about action. Building student ownership of mathematics requires both a plan *and* a sense of direction. I aim to ensure this book provides both. The intent of this closing section is to facilitate a sense of ownership in the reader and communicate that *You can do it.*

Each chapter will begin with its own overarching learning intention and set of specific success criteria to ground your learning by communicating our goals. Success criteria will have additional callouts throughout each chapter to model *signaling*, an aspect of teacher clarity discussed in the next chapter that helps guide learning by providing additional structure. Each chapter will conclude with reflection questions, to help you make personal connections to your own practices and mathematical experiences, as teacher clarity also encompasses understanding ourselves. Speaking of clarity—let's start there.

PART I

Determining Success Criteria

What Does It Mean to Be Successful in Mathematics?

1

Teacher clarity is more than a lesson plan; it's a sense of direction. Clarity allows instruction to be intentional and learning to be purposeful. After all, "*every student deserves a great teacher, not by chance but by design*" (Fisher et al., 2016). Pursuing clarity is the act of intentionally designing the great teacher that your students deserve. The argument for clarity, then, is simple: How can one expect to achieve any sort of outcome if that intended outcome is unknown? In other words, how do we expect to hit a target we aren't aiming for? If we are to make the largest learning gains with our students, and promote ownership of their learning and of the content itself, then we ought to begin by spending some time clarifying what it means to be successful in mathematics. What is it, exactly, that we should want for our students?

Teacher clarity is more than a lesson plan; it's a sense of direction

This chapter is dedicated to this concept of teacher clarity but specifically tailored for mathematics. I will begin by further defining teacher clarity and building the case for its pursuit—which is hard work! From there we will build consensus around what is meant by *success in mathematics* by compiling and examining existing works in the mathematics education community.

CHAPTER 1

Learning Intention:

I am pursuing teacher clarity by learning what it means to be successful in mathematics.

Success Criteria:

- ☐ I can explain the value of teacher clarity and the positive impact it has on students.
- ☐ I can identify connections between the Five Strands of Mathematical Proficiency, the Standards for Mathematical Practice, and the three aspects of mathematical rigor.
- ☐ I can define *success* in mathematics.

The Importance of Clarity and Measuring Success

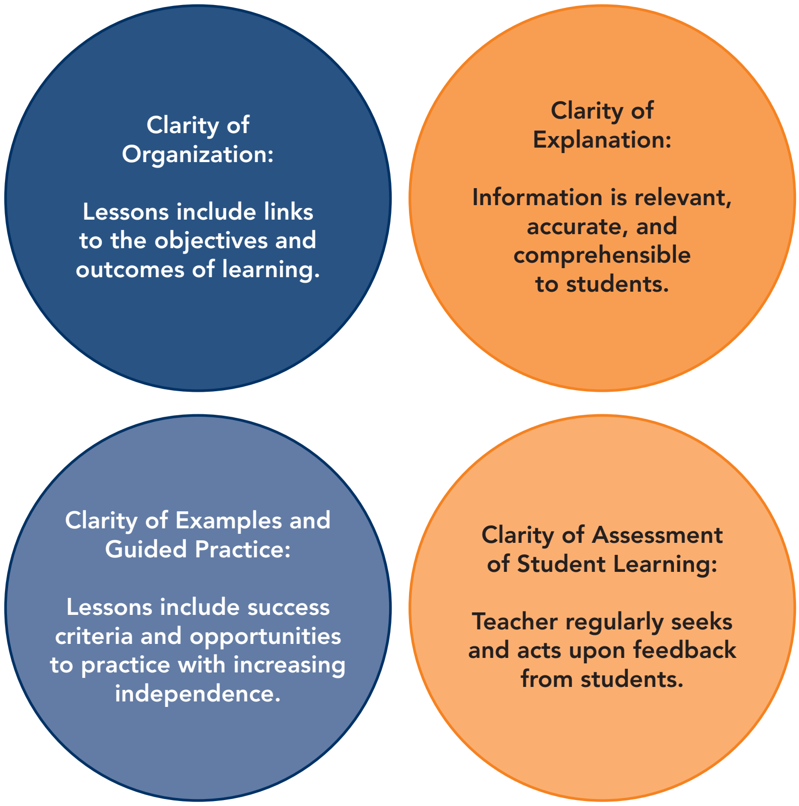
Teacher clarity: a measure of clarity of communication between teachers and students in both directions.

Teacher clarity is not a new concept. It has been studied and measured for the last half century in a variety of K–12 and college instructional settings. Fendick (1990) defined teacher clarity as "a measure of the clarity of communication between teachers and students in both directions" (p. 10). Fendick (1990) conducted a meta-analysis investigating teacher clarity across four dimensions, illustrated in Figure 1.1. The four dimensions that constitute his definition are (1) Clarity of Organization; (2) Clarity of Explanation; (3) Clarity of Examples and Guided Practice; and (4) Clarity of Assessment of Student Learning. Let's consider each of these.

Clarity of organization occurs at the lesson, the unit, and the whole year/whole-course level and includes such features as determining and stating learning intentions, aligning the content to formative and summative assessments, and reviewing content throughout the year/course.

Clarity of examples and guided practice refers to keeping instruction aligned to assessments, interacting formatively with students, providing time for practice, providing metrics of success (success criteria), and providing students with formative feedback (Fendick, 1990).

Figure 1.1 Four Dimensions of Teacher Clarity



Clarity of explanation is how the teacher simplifies or clarifies explanations and infuses them with relevance. It also refers to how a teacher emphasizes and reemphasizes directions and key points, connects content to prior knowledge, and pursues appropriate pacing based on student understanding and concept mastery.

Clarity of assessment of student learning includes the methods the teacher uses to check for understanding throughout a lesson, encourage class discussion, and provide feedback on assignments and assessments (Fendick, 1990).

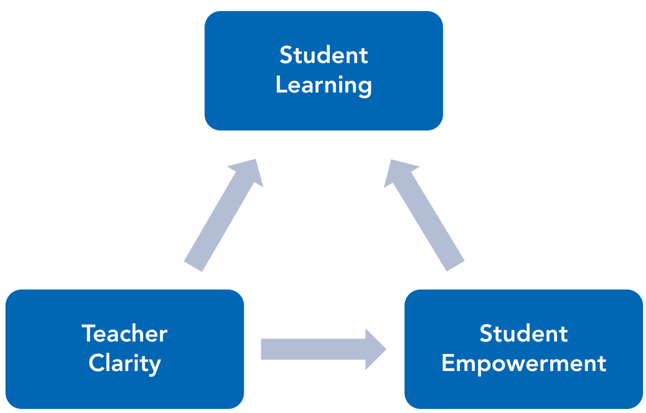
Everything that occurs during instruction should be intentionally linked toward some common end (what students need to learn)—and that common end should be understood by both teachers and students.

CHAPTER 1 SUCCESS CRITERIA CALLOUT:

- ☐ I can explain the value of teacher clarity and the positive impact it has on students.

Fendick (1990) was not the only researcher to determine that teacher clarity has a positive influence on student learning. Houser and Frymier (2009) studied how both student characteristics (namely, temperament and learner orientation) and teacher behaviors (nonverbal immediacy and clarity) influenced student empowerment. Both teacher immediacy and teacher clarity were found to have a greater effect on empowerment than any student characteristics measured in their study. Further, teacher clarity was found to have a direct impact on learning outcomes in addition to its indirect effect on outcomes through empowerment. This builds the case that teachers have the power to directly impact student empowerment, which is a precursor to efficacy. The evidence suggests that teachers have a greater impact on students' empowerment than students do themselves—which arguably reframes this task as a *responsibility* for teachers. Figure 1.2 demonstrates how teacher clarity has both a direct impact on student learning and an indirect impact on learning via student empowerment.

Figure 1.2 Teacher clarity has both a *direct* and *indirect* impact on student learning.

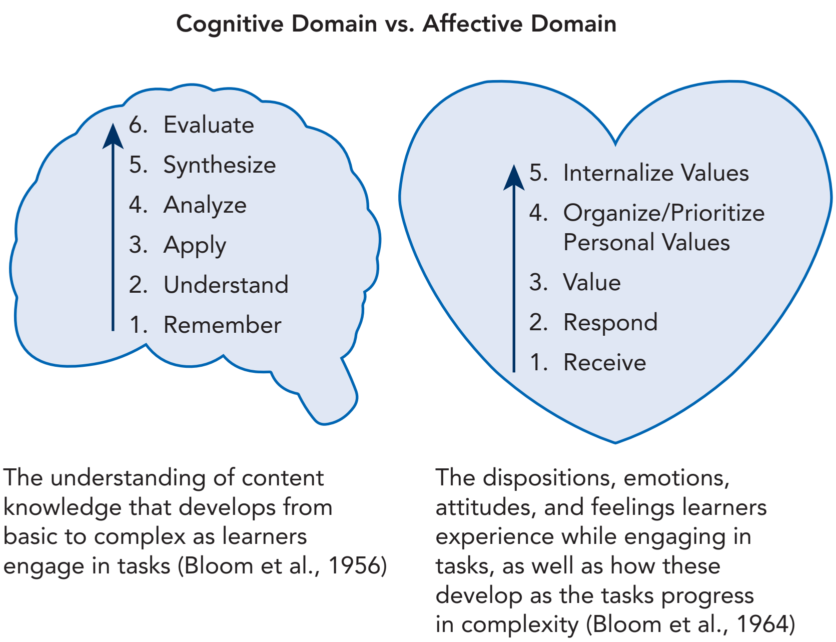


Further building this case, Titsworth et al. (2015) conducted two meta-analyses supporting the positive effects of teacher clarity on student learning. The results showed affective learning was impacted more than cognitive learning (200% increase versus

100% increase). Figure 1.3 further compares and contrasts the affective and cognitive domains of learning. This emphasis on affective learning would appear to align to Houser and Frymier's (2009) findings on teacher clarity's impact on student empowerment. Increases in teacher clarity also reduce the cognitive load of learning and increase motivation (Serki & Bolkan, 2024).

Signaling is another specific aspect of teacher clarity, which involves teaching with instructional and organizational cues that make the structure of the lesson transparent to students (Bolkan, 2017). In practice, this can commonly be seen as the interweaving and revisiting of learning intentions and success criteria throughout instruction. (You may have noticed this book making use of signaling with the Chapter Success Criteria Callouts.) For many students, signaling is an organizational and structural scaffold that frees up working memory for other tasks, such as learning the actual mathematics content of the lesson. These findings arguably situate teacher clarity as an equitable teaching issue.

Figure 1.3 Teacher Clarity's Impact on Student Empowerment



Source: Adapted from Bloom, et al., 1956 and Bloom, et al., 1964. Brain outline courtesy of iStock.com/Anatoliy Stepura

With the case being made for the benefits of teacher clarity, how might we actually begin implementing this measurable influence on learning? We should set our eyes on the prize: What do we and should we *intend* for our students? What does it mean to be successful in mathematics? The five strands of mathematical proficiency are key to answering these questions.

CHAPTER 1 SUCCESS CRITERIA CALLOUT:

- ☐ I can identify connections between the Five Strands of Mathematical Proficiency, the Standards for Mathematical Practice, and the three aspects of mathematical rigor.

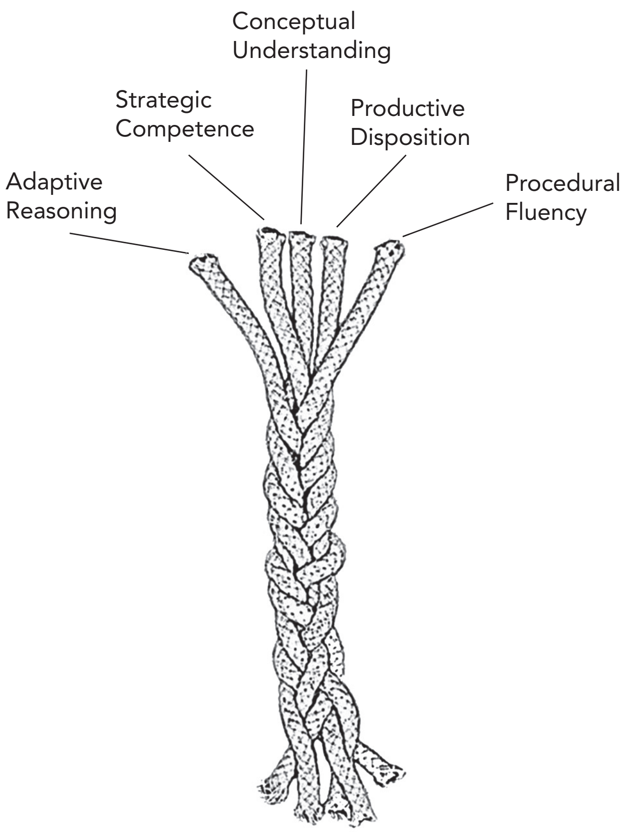
The Five Strands of Mathematical Proficiency

For some, success in school mathematics is simple to define: right answers signal success, while wrong answers signal failure. And while this perspective holds an obvious validity from a very literal point of view, research has expanded our view of success to include emphasis on both procedures and understanding. So if right and wrong answers are no longer the determining factors of success in mathematics, what measures do we replace them with? Enter the Five Strands of Mathematical Proficiency (Kilpatrick et al., 2001). The National Research Council defines *mathematical proficiency* as existing across five strands:

- ***Conceptual understanding***: comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency***: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence***: ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning***: capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition***: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence of one's own efficacy (Kilpatrick et al., 2001)

This work places special emphasis on the fact that understanding is greater than memorization, that the connections garnered by deep learning are prerequisite to transferring knowledge to novel situations, and that metacognition and motivation are both pivotal to learning. Kilpatrick et al. (2001) argue that, "[m]athematical proficiency . . . cannot be achieved by focusing on just one or two of these strands," but note that it is also not a simple dichotomy of either proficient or not (p. 116). So, mathematical proficiency should be developed along each strand, across the strands, and over time. The intertwined nature of these five strands is further illustrated in Figure 1.4. The remainder of this section will be dedicated to fleshing out each of these strands.

Figure 1.4 Five Intertwined Strands of Mathematical Proficiency



Source: Used with permission of The National Academies Press from *Adding it up: Helping children learn mathematics,* Kilpatrick et al., 2001; permission conveyed through Copyright Clearance Center, Inc.

Conceptual Understanding

Conceptual understanding manifests itself when students comprehend mathematical concepts, operations, and relations in a functional and integrated way (Kilpatrick et al., 2001). The connected nature of conceptual understanding organizes content and allows new knowledge to *fit*, rather than live in isolation. This has multiple benefits for students. For example, students are able to recognize new iterations of old concepts, or, "superficially unrelated situations" (different in appearance or context only), thus resulting in less to learn (Kilpatrick et al., 2001). Another benefit is that students recognize errors as they occur, largely because they don't *fit* existing expectations. This integration of knowledge also improves students' retention of information and breeds confidence by reinforcing their own sense of logical reasoning and natural conclusions. For example, when a student actually understands a concept, forgotten details can be logically reconstructed and checked against metrics of reason rather than just hoping that the information will pop back into their head somehow.

Sometimes proving challenging to measure, early conceptual understanding can exist in a learner prior to their ability to demonstrate it. Generally, however, conceptual understanding can be elicited through expressing connections between representations and concepts (via concept maps or other linking diagrams, explanations, etc.). Conceptual understanding can also manifest through a student's ability to produce multiple representations of the same mathematical situation, compare and contrast each, and on a deeper level recognize the contextual usefulness of each representation.

One common strategy for teaching and assessing conceptual understanding in mathematics is the Frayer Model (Frayer et al., 1969). Often undersold as a vocabulary instruction tool, the Frayer Model helps students flesh out the contours of concepts by considering the definition, examples of the concept, nonexamples, and a pictorial representation or other characteristics. As an assessment tool, Frayer Models can be provided without the term present in the center. Students are to read the definition, explore the characteristics, examples and nonexamples, and then try to determine which concept is being represented. Figure 1.5 shows both of these approaches.

Procedural fluency is about possessing procedural skills and a sense of direction about when to use them, as well as a certain degree of automaticity

Procedural Fluency

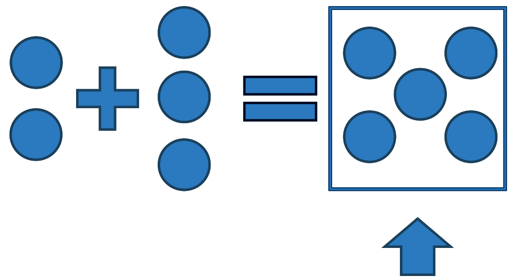
Procedural fluency is about possessing procedural skills and a sense of direction about when to use them, as well as a certain degree of automaticity (Kilpatrick et al., 2001). In addition to promoting students' mathematical independence, procedural fluency provides students a lens into the well-structured nature of mathematics that can be generative toward their own inquiry. In other words, a high degree of procedural fluency provides students with an ability to conduct tests and experiments within and upon mathematics.

Figure 1.5a Frayer Model Example 1

Definition:

The result from addition.

Image or Characteristics:



Concept:

Sum

Examples:

3 + 2 = 5

7 = 5 + 2

$$\begin{array}{r} 87 \\ +\ 21 \\ \hline 108 \end{array}$$

Non-examples:

10 – 3 = 7

3 × 4 = 12

100 ÷ 10 = 10

Figure 1.5b Frayer Model Example 2

Definition:

The set of numbers that includes whole numbers and their opposites.

Image or Characteristics:

Has no fractional or decimal parts

- Can be positive
- Can be negative
- Can be zero
- Can be modeled with two color tiles

Concept:

Examples:

–7	12	–9,878
1	–1	1,000,000
0	400	

Non-examples:

1.3	1/2	–1.754
π	–43	$3\frac{1}{4}$
–5/4		

Fluency also frees up working memory for students to engage in the other strands of mathematical proficiency by easing the difficult and simplifying the complex through familiarity.

The National Council of Teachers of Mathematics (2014) argues through their Mathematics Teaching Practices that procedural fluency must be built on a foundation of conceptual understanding. Further, teaching isolated procedures first can make students resistant to investigating their conceptual underpinnings (Kilpatrick et al., 2001). Also, rooting procedures in conceptual understanding is more efficient and will require less commitment to memory and less practice toward retention, which again speaks to the connected nature of conceptual understanding (Kilpatrick et al., 2001). As teachers, this means that we should not choose *between* teaching concepts or procedures but instead should thoughtfully sequence them.

Strategic Competence

Strategic competence goes beyond problem solving to include two of its precursors, formulating mathematical problems and representing them (Kilpatrick et al., 2001). This strand encompasses much of what would be considered mathematical modeling, as well as the related process of mathematizing the real world or realistically imaginable (Gravemeijer & Doorman, 1999) and speaks to the field of applied mathematics. While mathematics is indeed the universal language, it is rarely spelled out for us in the real world. This is why students need to be able to understand situations and their key features and then mathematize the relevant features while ignoring those that are irrelevant to the problem or situation (Kilpatrick et al., 2001). This mathematical modeling process could involve crafting an equation or some other representation of the situation, such as a drawing or diagram. For instance, elementary students might be tasked with calculating the area of sports fields or other grassy areas by mathematizing an aerial view of their actual shapes into composites of known two-dimensional shapes. Secondary students might be tasked with measuring the heights of buildings and other tall objects using an inclinometer and trigonometry. Broadening students' exposure from simply routine problems to also include nonroutine problems such as these that require productive and inventive thinking builds flexibility with strategic competence. Students develop further fluency through strategic competence when they translate their hindsight gained from a nonroutine problem into foresight the next time they engage in a similar task.

Adaptive Reasoning

Adaptive reasoning is possibly the most challenging strand to unwind from the others, as its impact is absolutely pervasive throughout all mathematics. Adaptive reasoning speaks to the logical connections between mathematical ideas and one's ability to communicate those connections. This strand is from where all mathematicians gain their authority by borrowing it from the purity of the subject itself. Adaptive reasoning empowers students to be their own metric of success by tapping into their own sense of reason. Adaptive reasoning is a broad idea that encompasses both formal and informal modes of communication, such as proofs, oral discussion, informal explanation and justification, and intuitive reasoning based on patterns, analogies, metaphors, and other thinking tools. Thus, adaptive reasoning is measurable through students' ability to justify their work and explain their thinking. And while any success criteria aimed at measuring adaptive reasoning should absolutely be operationalized at the individual level, this area also begs the use of social and sociomathematical norms, which will be discussed in the next chapter.

Productive Disposition

Productive disposition is about learners recognizing the value of and thus caring about mathematics, while also believing that they have the capacity to learn it. Students who have a productive disposition toward mathematics believe that the subject is understandable, it should make sense, and, with appropriate effort, is conquerable and worthwhile. Most children begin their schooling with a productive disposition toward mathematics; however, this often changes for many somewhere along their academic careers depending on their experiences (Kilpatrick et al., 2001). Further, students' experiences shape their beliefs about mathematics and hence their dispositions toward the subject. This situates teachers in a very powerful and consequential position to shape *and maintain* students' productive dispositions toward mathematics.

Productive disposition is about learners recognizing the value of and thus caring about mathematics, while also believing that they have the capacity to learn it.

The Standards for Mathematical Practice

While most standards documents tend to delineate the various concepts and procedures to be gained through the content standards of each course, many have also begun fleshing out additional practice standards that embody the latter three strands of mathematical proficiency. Practice standards are a wonderful addition for this reason. They provide a space to explicitly call out the habits of mind, dispositions, and approaches identified in the Strands of Mathematical Proficiency that would otherwise only be implicit (at best) in content standards. These practice standards should not be viewed as additive but rather as clarifying for our task as mathematics teachers. Consider the alignment in Table 1.1 between the eight Standards for Mathematical Practice (National Governors Association Center for Best Practice, Council of Chief State School Officers, 2010) and the five Strands of Mathematical Proficiency.

Table 1.1 Alignment Between the Standards for Mathematical Practice and Five Strands of Mathematical Proficiency

STANDARD FOR MATHEMATICAL PRACTICE	RELATED STRANDS OF MATHEMATICAL PROFICIENCY
1. Make sense of problems and persevere in solving them.	Productive Disposition
2. Reason abstractly and quantitatively.	Adaptive Reasoning; Strategic Competence
3. Construct viable arguments and critique the reasoning of others.	Adaptive Reasoning
4. Model with mathematics.	Strategic Competence
5. Use appropriate tools strategically.	Strategic Competence
6. Attend to precision.	Adaptive Reasoning
7. Look for and make use of structure.	Adaptive Reasoning; Strategic Competence
8. Look for an express regularity in repeated reasoning.	Adaptive Reasoning; Strategic Competence

Rigor: the balanced pursuit of conceptual understanding, procedural skills and fluency, and application with equal intensity

Three Aspects of Rigor in Mathematics

Rigor is the balanced pursuit of conceptual understanding, procedural skills and fluency, and application with equal intensity (Common Core State Standards Initiative [CCSSI], 2020). To more fully understand how rigor is reflected in content standards, let's look at each aspect of the definition.

- **Conceptual understanding:** The standards call for conceptual understanding of key concepts, such as place value and ratios.

Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.

- **Procedural skills and fluency:** The standards call for speed and accuracy in calculation. Students must practice core functions, such as single-digit multiplication, in order to have access to more complex concepts and procedures. Fluency must be addressed in the classroom or through supporting materials, as some students might require more practice than others.
- **Application:** The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency.

In order to have clarity around instructional design and sequencing, it is important to recognize and understand these three aspects of rigor both independently and interdependently. Some standards, for instance, call for each of these aspects of rigor in isolation. For example, consider the following second-grade standard regarding number and operations in base ten:

> 2.NBT.1 Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens—called a "hundred." b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). (Taken from CA Mathematics standards)

By use of the verb *understand*, it is clear that this standard is specifically targeting a student's conceptual understanding of place value. No action other than understanding is called out and thus expected of students. Teachers would do well to recognize this and plan instruction with this target in mind.

Similarly, consider this sixth-grade standard regarding number sense and calculations with decimals:

> 6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation. (Taken from CA Mathematics standards)

By similar investigation of the verbs in the standard and actions expected of students—namely to *fluently add, subtract, multiply, and divide*, as well as to *use the standard algorithm*, it is clear that this standard is aiming to build students' procedural skills and fluency. This recognition would serve teachers well when identifying the learning intentions and determining what success would look like for this standard.

Finally, consider this high school integrated Math II standard regarding geometric measurement and dimension:

> G-GMD.6 Verify experimentally that in a triangle, angles opposite longer sides are larger, sides opposite larger angles are longer, and the sum of any two side lengths is greater than the remaining side length; apply these relationships to solve real-world and mathematical problems. (Taken from CA Mathematics standards)

The use of the term "verify experimentally" signals that students should be engaging in hands-on experiences, and "real-world" makes it clear that the goal of this standard is for students to apply their mathematics to novel situations.

Notably, this final standard is asking students to apply their procedural skills and conceptual understanding around solving side and angle measurements for triangles. This should not be understated as many standards documents state that, "Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural fluency" (CCSSI, 2020). To state this plainly, conceptual understanding is a prerequisite to procedural skills and fluency, and both are prerequisite to application. While application is indeed the leveraging of both conceptual understanding and procedural skills, this does not isolate it to the final phases of instruction. Application can be used throughout a unit of study to discover the need for new procedures or to provide access to a concept through real-world familiarity (viral videos, purchasing things from a menu, etc.). It also exists as the ultimate display of mathematical ownership of given concepts and skills. Figure 1.6 illustrates this balanced approach to mathematics teaching and learning.

In stark contrast to this balanced approach, the traditional math classrooms that many of us experienced as students ourselves were almost entirely focused on procedures (see Figure 1.7). The idea here was to arm us all with the tools we would use later in calculus. (How is *that* for

Figure 1.6 A Balanced Approach to Mathematics Instruction

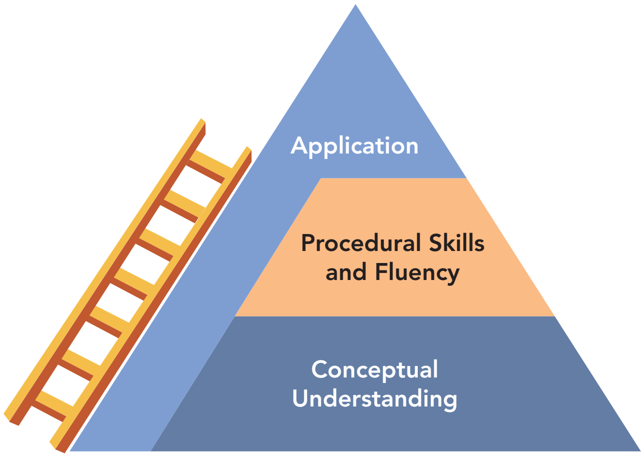
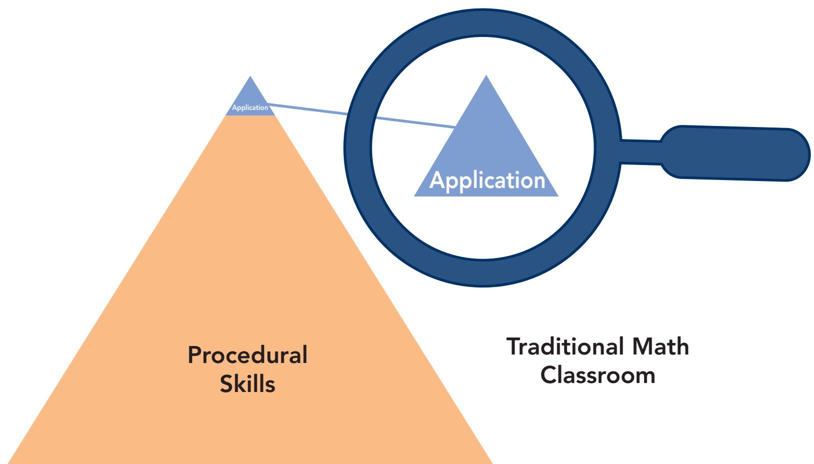


Figure 1.7 An Unbalanced Approach to Mathematics Instruction in a Traditional Math Classroom



a faith-based approach to relevance? *Learn this—I promise you'll need it later!*) Application was saved for the end of each section or chapter *if we had time*—which meant it was often cut. An important reckoning for some of us, however, is that we accidentally developed a conceptual understanding along the way. As teachers of math now ourselves, we must be careful not to give more credit than is due to the system that built us. We didn't learn math and develop a sense of ownership *because* of the traditional approach; we did so *despite* it.

Determining Our Success Criteria

So, how is it that a student develops a productive disposition, conceptual understanding, procedural fluency, and leverages those toward strategic competence, adaptive reasoning, and application—all the while exhibiting the habits of mind and practices of a mathematician? This task requires students to take *ownership* of the content and situate themselves as active practitioners. Ultimately, then, success in mathematics is the propensity to learn and use as much of it as one needs or wants. This is mathematical ownership. When asked *Whose math is it?* successful students will respond *my math!*

CHAPTER 1 SUCCESS CRITERIA CALLOUT:

- ☐ I can define *success* in mathematics.

If this is what it means for students to be successful in mathematics, then our task as mathematics educators is to create spaces that promote productive dispositions and set the stage for strategic competence, to interact in ways that develop agency, and to create experiences that foster increasing independence and adaptive reasoning. This is the vehicle and setting through which we will be able to teach toward a balance of conceptual understanding, procedural fluency, and application. *Clarity achieved.* Now that we have a goal in mind we can begin taking steps toward reaching it. The second part of this book will seek to do exactly that.

CHAPTER 1 REFLECTION QUESTIONS

- What are some examples from your classroom of students' productive dispositions changing over the course of a year? What might the catalysts have been for these changes?
- What does the balance of conceptual understanding, procedural skills and fluency, and application look like in your classroom? How does this compare to your own experience as a math student?
- As you consider the Strands of Mathematical Proficiency, where are you finding opportunities to build student ownership across each?