CHAPTER ONE

Introduction

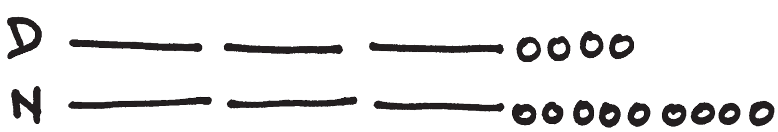
Why You Need to Teach Students to Mathematize

Imagine you are a new teacher. You are teaching second grade at a new school and are eager to get to know your students—their interests, skills, and how prepared they are to meet the challenges of second grade. You have just emerged from your teacher education program knowing various approaches you have seen modeled in classrooms and described in the literature, some of which you have tried with varying degrees of success. You aren't sure what approaches you want to use but are excited about challenging your students, introducing the rigor you have read so much about. But first, you need to know what your students can and can't do.

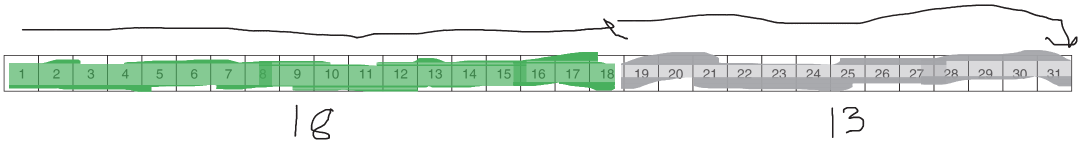
You decide to start with a couple of word problems, ones that involve friendly numbers and relatively simple mathematical operations:

> *Daphne has 35 shells in her collection, 4 more than Nathan. How many shells does Nathan have?*
>
> *Raphine had 18 books. He bought 13 more at the library book sale. How many books does Raphine have now?*

You circulate around the room, noting who draws pictures, who writes equations, and who uses the manipulatives you have put at the center of the table groups. While some students take their time, quite a few move quickly. Their hands go up eagerly, indicating they have solved the problems. As you check their work, one by one, you notice most of them got the first problem wrong, writing the equation $35 + 4 = 39$. Some even include a sentence saying, "Nathan has 39 shells." Only one student in this group draws a picture. It looks like this:



Even though the second problem includes regrouping, a potentially complicating feature, most of these same students solve it correctly. They wrote the equation $18 + 13 = 31$ and were generally able to find the correct solution of 31 books in Raphine's collection. You notice a few students used the base 10 blocks available at tables to help them solve these problems. A number of the students used a number track that looked something like this:



To learn more about how students went wrong with the first problem, you call them to your desk one by one and ask about their thinking. A pattern emerges quickly. All of the students you talk to quickly zeroed in on two key elements of the problem: (1) the total quantity of shells Daphne has and (2) how many more Nathan has. One student tells you, "*More* always means to add. I learned that a long time ago." Clearly, she wasn't the only student who read the word *more* and assumed she had to add. This assumption, which led students astray in the first problem, luckily worked for these students in the second problem, where simple addition yielded a correct answer.

Problem-Solving Strategies Gone Wrong

In our work with teachers, we often see students being taught a list of "key words" that are linked to specific operations. Students are told, "Find the key word and you will know whether to add, subtract, multiply, or divide." Charts of key words often hang on classroom walls. Focusing on key words is a strategy that works often enough that teachers continue to rely on it. As we have seen in the shell problem, though, not only are key words not enough to solve a problem, but they can easily lead students to the wrong operation or to a single operation when other operations need to come into play (Karp, Bush, & Dougherty, 2014). As the shell problem reveals, students can call upon different strategies and operations depending on how they approach the problem—that is, counting on from an estimated value, or subtraction. The simple one-step addition operation we saw in the student's solution to the first problem, the one the student associated with the key word *more*, does not lead to a correct answer.

Let's return to your imaginary classroom. Having seen firsthand the limitations of key words—a strategy you had considered using—where to begin? What approach to use? A new colleague has a suggestion. She agrees that relying on only key words can be too limiting. Instead, she is an enthusiastic proponent of a procedure called CUBES, which stands for teaching students these steps:

Circle the numbers

Underline important information

Box the question

Eliminate unnecessary information

Solve and check

She tells you that whenever she introduces a new kind of word problem, she walks students through the CUBES protocol using a "think-aloud," sharing how she is using the process to take the problem apart to find what to focus on. That evening, as you settle down to plan, you decide to walk through some problems like the shell problem using CUBES. Circling the numbers is easy enough. You circle 35 (shells) and 4 (the difference between Daphne and Nathan).

Then you tackle "important information." What is important here in this problem? Daphne has 35 shells is important. Certainly the fact that this is 4 more than Nathan is important, but unfortunately that phrase includes the problematic word *more*. You box the question and realize there's not an action verb in this problem. Based on the student work and thinking you see, this will be challenging for many students to act out.

If you think this procedure has promise as a way to guide students through an initial reading of the problem, but leaves out how to help students develop a genuine understanding of the problem, you would be correct.

What is missing from procedural strategies such as CUBES and strategies such as key words, is—in a word—*mathematics*, and the understanding of where it lives within the situation the problem is presenting. Rather than helping students to learn and practice quick ways to enter a problem, we need to focus our instruction on helping them develop a deep understanding of the mathematical principles behind the operations and how they are expressed in the problem. They need to learn to *mathematize*.

What Is Mathematizing? Why Is It Important?

Mathematizing is the uniquely human process of constructing meaning in mathematics (from Freudenthal, as cited in Fosnot & Dolk, 2001). Meaning is constructed and expressed by a process of noticing, exploring, explaining, modeling, and convincing others of a mathematical argument. When we teach students to mathematize, we are essentially teaching them to take their initial focus off specific numbers and computations and put their focus squarely on the actions and relationships expressed in the problem, what we will refer to throughout this book as the **problem situation**. At the same time, we are helping students see how these various actions and relationships can be expressed and the different operations that can be used to express them. If students understand, for example, that comparison problems, like the shell problem, involve two quantities and the size of the gap between them, then they can learn where and how to use the values in the problem to create a number sentence. If we look at problems this way, then finding a **solution** involves connecting the problem's context to its general kind of problem situation and to the operations that go with it. The rest is simple computation.

Making accurate and meaningful connections between different problem situations and the operations that can fully express them requires **operation sense**. Students with a strong operation sense

- Understand and use a wide variety of models of operations beyond the basic and **intuitive model of an operation** (Fischbein, Deri, Nello, & Marino, 1985)
- Use appropriate representations of actions or relationships strategically
- Apply their understanding of operations to any quantity, regardless of the class of number
- Can mathematize a situation, translating a contextual understanding into a variety of other mathematical representations

Mathematizing: The uniquely human act of modeling reality with the use of mathematical tools and representations.

Problem situation: The underlying mathematical action or relationship found in a variety of contexts. These are often called "problem type" for short.

Solution: A description of the underlying problem situation along with the computational approach (or approaches) to finding an answer to the question.

Operation sense: Knowing and applying the full range of work for mathematical operations (for example, addition and subtraction).

Intuitive model of an operation: An intuitive model is "primitive," meaning that it is the earliest and strongest interpretation of what an operation, such as multiplication, can do. An intuitive model may not include all the ways that an operation can be used mathematically.

FOCUSING ON OPERATION SENSE

Many of us may assume that we have a strong operation sense. After all, the four operations are the backbone of the mathematics we were taught from day one in elementary school. We know how to add, subtract, multiply, and divide, don't we? Of course we do. But a closer look at current standards reveals nuances and relationships within these operations that many of us may not be aware of, may not fully understand, or may have internalized so well that we don't recognize we are applying an understanding of them every day when we ourselves mathematize problems both in real life and in the context of solving word problems. For example, current standards ask that students develop conceptual understanding and build procedural fluency in four kinds of addition and subtraction problems, including Add-To, Take-From, Compare, and what some call Put Together/Take Apart (we will refer to this category throughout the book as Part-Part-Whole). On the surface, the differences between such categories may not seem critical. But we argue that they are. Only by exploring these differences and the relationships they represent can students develop the solid operation sense that will allow them to understand and mathematize word problems and any other problems they are solving, whatever their grade level or complexity of the problem. But just as important, word problems offer the unique opportunity to engage in such exploration. Operation sense is not simply a means to an end. It has value in helping students naturally come to see the world through a mathematical lens.

USING MATHEMATICAL REPRESENTATIONS

What would such instruction—instruction aimed at developing operation sense and learning how to mathematize word problems—look like? It would have a number of features. First, it would require that we give students time to focus and explore by doing fewer problems, making the ones they do count. Next, it would facilitate students becoming familiar with various ways to represent actions and relationships presented in a **problem context**. We tend to think of solving word problems as beginning with words and moving toward number sentences and equations in a neat linear progression. But this isn't how problem solving works. It is an iterative and circular process, where students might try out different representations, including going back and rewording the problem, a process we call telling "the story" of the problem. The model that we offer in this book is based on this kind of active and expanded exploration using a full range of **mathematical representations**. Scholars who study mathematical modeling and problem solving identify five modes of representation: verbal, contextual, concrete, pictorial, and symbolic (Lesh, Post, & Behr, 1987).

Problem context: The specific setting for a word problem.

Mathematical representations: Depictions of a mathematical situation using one or more of these modes or tools: concrete objects, pictures, mathematical symbols, context, or language.

VERBAL A problem may start with any mode of representation, but a word problem is first presented verbally, often in written form, and may be read aloud for young learners. After that, verbal representations can serve many uses as students work to understand the actions and relationships in the problem situation. Some examples are restating the problem; thinking aloud; describing the math operations in words rather than symbols; and augmenting and explaining visual and physical representations such as graphs, drawings, counters, base 10 blocks, or other concrete items. Verbal representations do not have to be written.

CONTEXTUAL The contextual representation is simply the real-life situation that the problem describes. Prepackaged word problems are based on real life, as are the shell and book problems described earlier in this chapter, but alone they are not contextual. Asking students to create their own problems based on real-life contexts will bring more meaning to the process and will reflect the purposes of mathematics in real life, such as when scientists, business analysts, and meteorologists mathematize contextual information in order to make predictions that benefit us all. This is a process called **mathematical modeling** (Garfunkel & Montgomery, 2016).

Mathematical modeling: A process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into real-world phenomena.

CONCRETE Using physical representations such as blocks, concrete objects, and real-world items (for example, money, measuring tools, or items to be measured such as beans, sand, or water), or acting out the problem in various ways, is called **modeling**. Such concrete models often offer the closest and truest representation of the actions and relationships in a problem situation.

Modeling: Creating a physical representation of a problem situation.

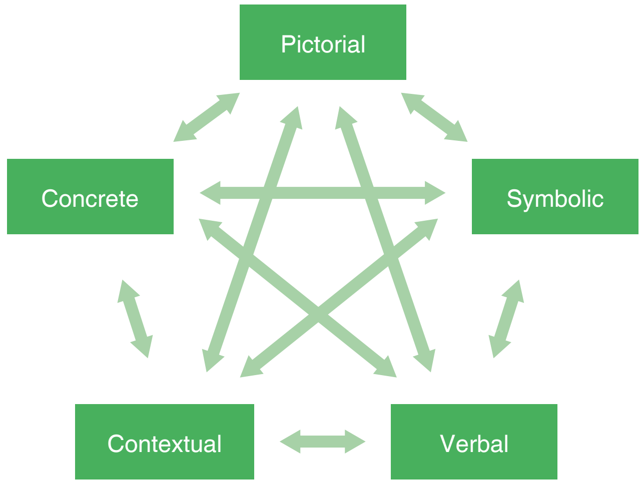
PICTORIAL Pictures and diagrams can illustrate and clarify the details of the actions and relationships in ways that words and even physical representations cannot. Using dots and sticks, bar models, arrows to show action, number lines, boxes to show regrouping, and various graphic organizers helps students see and conceptualize the nature of the actions and relationships.

SYMBOLIC Symbols can be operation signs (+, –, ×, ÷), relational signs (=, <, >), variables or unknowns (typically expressed first with a box, space, or question mark and later as *x, y, a,* or *b*), or a wide variety of symbols used in later mathematics (k, ∞, ϕ, π, etc.). Even though numerals are familiar, they are also symbols representing a value (2, 0.9, $\frac{1}{2}$, 1000).

There are two things to know about representations that may be surprising. First, mathematics can be shared *only* through representations. As a matter of fact, it is impossible to share a mathematical idea with someone else without sharing it through a representation! If you write an equation, you have produced a *symbolic* representation. If you describe the idea, orally or in writing, you have shared a *verbal* representation. Representations are not solely the manipulatives, pictures, and drawings of a mathematical idea: They are any mode that communicates a mathematical idea between people.

Second, the strength and value of learning to manipulate representations to explore and solve problems is rooted in their relationship to one another. In other words, the more students can learn to move deftly from one representation to another, translating and/or combining them to fully illustrate their understanding of a problem, the deeper will be their understanding of the operations. Figure 1.1 reveals this interdependence. The five modes of representation are all equally important and deeply interconnected, and they work synergistically. In the chapters that follow, you will see how bringing multiple and synergistic representations to the task of problem solving deepens understanding.

FIGURE 1.1 FIVE REPRESENTATIONS: A TRANSLATION MODEL



Source: Adapted from Lesh, Post, and Behr (1987).

Teaching Students to Mathematize

As we discussed earlier, learning to mathematize word problems to arrive at solutions requires time devoted to exploration of different representations with a focus on developing and drawing on a deep understanding of the operations. We recognize that this isn't always easy to achieve in a busy classroom; hence, the appeal of the strategies we mentioned at the beginning of the chapter. But what we know from our work with teachers and our review of the research is that, although there are no shortcuts, structuring exploration to focus on actions and relationships is both essential and possible. Doing so requires three things:

1. Teachers draw on their own deep understanding of the operations and their relationship to different word problem situations to plan instruction.
2. Teachers use a model of problem solving that allows for deep exploration.
3. Teachers use a variety of word problems throughout their units and lessons, to introduce a topic and to give examples during instruction, not just as the "challenge" at the end of the chapter.

In this book we address all three.

BUILDING YOUR UNDERSTANDING OF THE OPERATIONS AND RELATED PROBLEM SITUATIONS

The chapters that follow explore the different operations and the various kinds of word problems—or problem situations—that arise within each. To be sure that all of the problems and situational contexts your students encounter are addressed, we drew on a number of sources, including the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010); the work done by the Cognitively Guided Instruction projects (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Franke, Levi, & Empson, 2014); earlier research; and our own work with teachers to create tables, one for addition and subtraction situations (Figure 1.2) and another for multiplication and division situations (Figure 1.3). Our versions of the problem situation tables represent the language we have found to resonate the most with teachers and students as they make sense of the various problem types, while still accommodating the most comprehensive list of categories. These tables also appear in the Appendix at the end of the book.

NOTES

FIGURE 1.2 ADDITION AND SUBTRACTION PROBLEM SITUATIONS

ACTIVE SITUATIONS

	Result Unknown	Change Addend Unknown	Start Addend Unknown
Add-To	Paulo counted 9 crayons. He put them in the basket. Paulo found 6 more crayons under the table. He put them in the basket. How many crayons are in the basket? $9+6=x$ $6=x-9$	Paulo counted 9 crayons. He found more and put them in the basket. Now Paulo has 15 crayons. How many crayons did he put in the basket? $9+x=15$ $9=15-x$	Paulo had some crayons. He found 6 more crayons under the table. Now he has 15 crayons. How many crayons did Paulo have in the beginning? $x+6=15$ $15-6=x$
Take-From	There are 19 students in Mrs. Amadi's class. 4 students went to the office to say the Pledge. How many students are in the class now? $19-4=x$ $4+x=19$	There are 19 students in Mrs. Amadi's class. Some students went to class to read the Pledge. There were still 15 students in the classroom. How many students went to the office? $19-x=15$ $x+15=19$	4 students went to the office. 15 students were still in the classroom. How many students are there in Mrs. Amadi's class? $x-4=15$ $15+4=x$

RELATIONSHIP (NONACTIVE) SITUATIONS

	Total Unknown	One Part Unknown	Both Parts Unknown
Part-Part-Whole	The first grade voted on a game for recess. 11 students voted to play four square. 8 voted to go to the playground. How many students are in the class? $8+11=x$ $x-11=8$	The 19 first graders voted on a recess activity. 8 students voted to go to the playground. How many wanted to play four square? $8+x=19$ $x=19-8$	The 19 first graders voted on a recess activity. Some wanted to play four square. Some wanted to go to the playground. What are some ways the first graders could have voted? $x+y=19$ $19-x=y$

	Difference Unknown	Greater Quantity Unknown	Lesser Quantity Unknown
Additive Comparison	Jessie's paper airplane flew 14 feet. Jo's paper airplane flew 9 feet. How much less did Jo's paper airplane fly than Jessie's? $14-9=x$ $9+x=14$	Jo's paper airplane flew 9 feet. Jessie's paper airplane flew 5 feet more than Jo's. How far did Jessie's paper airplane fly? $9+5=x$ $x-5=9$	Jessie's paper airplane flew 14 feet. Jo's paper airplane flew 5 feet less than Jessie's paper airplane. How far did Jo's paper airplane fly? $14-5=x$ $14=x+5$

Note: The representations for the problem situations in these tables reflect our understanding based on a number of resources. These include the tables in the Common Core State Standards for Mathematics (CCSS-M; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), the problem situations as described in the Cognitively Guided Instruction research (Carpenter, Hiebert, & Moser, 1981), and other tools. See the Appendix and the book's companion website (resources.corwin.com/problemsolvingk-2) for a more detailed summary of the documents that informed our development of these tables.

FIGURE 1.3 MULTIPLICATION AND DIVISION PROBLEM SITUATIONS

ASYMMETRICAL (NONMATCHING) FACTORS				
	Product Unknown	**Multiplier (Number of Groups) Unknown**	**Measure (Group Size) Unknown**	
Equal Groups	Mayim has 8 vases to decorate the tables at her party. She places 2 flowers in each vase. How many flowers does she need? $8 \times 2 = x$ $x \div 8 = 2$	Mayim has some vases to decorate the tables at her party. She places 2 flowers in each vase. If she uses 16 flowers, how many vases does she have? $x \times 2 = 16$ $x = 16 \div 2$	Mayim places 16 flowers in vases to decorate the tables at her party. There are 8 vases and each vase has the same number of flowers. How many flowers will be in each vase? $8 \times x = 16$ $16 \div 8 = x$	
	Resulting Value Unknown	**Scale Factor (Times as Many) Unknown**	**Original Value Unknown**	
Multiplicative Comparison	Amelia's dog is 5 times older than Wanda's 3-year-old dog. How old is Amelia's dog? $5 \times 3 = x$ $x \div 5 = 3$	Sydney has $15 to spend at the movies. Her sister has $5. How many times more money does Sydney have than her sister has? $x \times 5 = 15$ $5 = 15 \div x$	Mrs. Smith has 15 puzzles in her classroom. That is 3 times as many puzzles as are in Mr. Jackson's room. How many puzzles are in Mr. Jackson's room? $3 \times x = 15$ $15 \div 3 = x$	

SYMMETRICAL (MATCHING) FACTORS			
	Product Unknown	**One Dimension Unknown**	**Both Dimensions Unknown**
Area/Array	Bradley bought a new rug for the hallway in his house. One side measured 5 feet and the other side measured 3 feet. How many square feet does the rug cover? $5 \times 3 = x$ $3 + 3 + 3 + 3 + 3 = x$ $3 \times 5 = x$ $5 + 5 + 5 = x$	The 12 members of the student council lined up on the stage to take yearbook pictures. The first row started with 6 students and the rest of the rows did the same. How many rows were there? $6 \times x = 12$ $x = 12 \div 6$	Daniella was building a house foundation using her building blocks. She started with 20 blocks. How many blocks long and wide could the foundation be? $x \times y = 20$ $20 \div x = y$
	Sample Space (Total Outcomes) Unknown	**One Factor Unknown**	**Both Factors Unknown**
Combinations (Fundamental Counting Principle)	Karen has 3 shirts and 7 pairs of pants. How many unique outfits can she make? $3 \times 7 = x$ $3 = x \div 7$	Evelyn says that she can make 21 unique and different ice cream sundaes using just ice cream flavors and toppings. If she has 3 flavors of ice cream, how many kinds of toppings does Evelyn have? $3 \times x = 21$ $21 \div 3 = x$	Audrey can make 21 different fruit sodas using the machine at the restaurant. How many different flavorings and sodas could there be? $x \times y = 21$ $x = 21 \div y$

Note: In the upper elementary grades, students begin the long journey of learning to think multiplicatively and proportionally. Part of this process involves moving away from counting and repeated addition to represent ideas that are better expressed with multiplication, but the primary years are still focused mostly on counting and adding. Some standards leverage that strength to introduce early ideas of multiplication: Counting squares in an array is one of them, and skip counting is another. We have included multiplication and division equations for our adult readers. K–2 students are not typically expected to represent these operations in equation form.

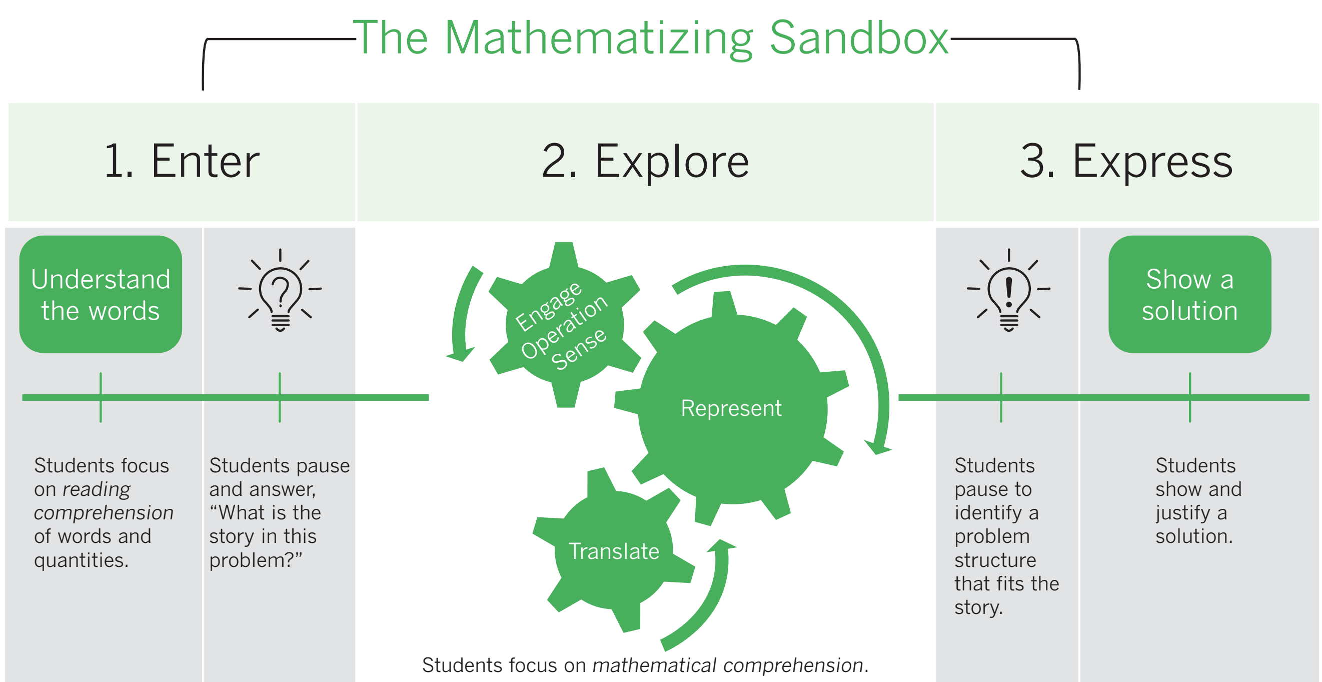
In the chapters—each of which corresponds to a particular problem situation and a row on one of the tables—we walk you through a problem-solving process that enhances your understanding of the operation and its relationship to the problem situation while modeling the kinds of questions and explorations that can be adapted to your instruction and used with your students. Because this volume emphasizes the work of grades K–2, we begin with a discussion about counting and then look deeply at each situation addressed by addition and subtraction. We end with a chapter introducing the early ideas of multiplication and division that have their roots in primary mathematics.

In each chapter, you will have opportunities to stop and engage in your own problem solving in the workspace provided. We end each chapter with a summary of the key ideas for that problem situation and some additional practice that can also be translated to your instruction.

PLAYING IN THE MATHEMATIZING SANDBOX: A PROBLEM-SOLVING MODEL

To guide your instruction and even enhance your own capacities for problem solving, we have developed a model for solving word problems that puts the emphasis squarely on learning to mathematize (Figure 1.4). The centerpiece of this model is what we call the "mathematizing sandbox," and we call it this for a reason. The sandbox is where children explore and learn through play. Exploring, experiencing, and experimenting by using different representations is vital not only to developing a strong operation sense but also to building comfort with the problem-solving process. Sometimes it is messy and slow, and we as teachers need to make room for it. We hope that the model illustrated in the figure will be your guide.

FIGURE 1.4 A MODEL FOR MATHEMATIZING WORD PROBLEMS



The mathematizing sandbox involves three steps and two pauses:

Step 1 (Enter): Students' first step is one of reading comprehension. Students must understand the words and context involved in the problem before they can really dive into mathematical understanding of the situation, context, quantities, or relationship between quantities in the problem.

Pause 1: This is a crucial moment when, rather than diving into an approach strategy, students make a conscious choice to look at the problem a different way, with a mind toward reasoning and sense-making about the *mathematical story* told by the problem or context. You will notice that we often suggest putting the problem in your own words as a way of making sense. This stage is critical to moving beyond the "plucking and plugging" of numbers with no attention to meaning that we so often see (SanGiovanni, 2020).

Step 2 (Explore): We call this phase of problem solving "stepping into the mathematizing sandbox." This is the space in which students engage their operation sense and play with some of the different representations mentioned earlier, making translations between them to truly understand what is going on in the problem situation. What story is being told? What are we comparing, or what action is happening? What information do we have, and what are we trying to find out? This step is sometimes reflected in mnemonics-based strategies such as STAR (stop, think, act, review) or KWS (What do you know? What do you want to know? Solve it.) or Pólya's (1945) four steps to problem solving (understand, devise a plan, carry out a plan, look back) or even CUBES. But it can't be rushed or treated too superficially. Giving adequate space to the explore phase is essential to the understanding part of any strategic approach. This is where the cognitive sweet spot can be found, and this step is what the bulk of this book is about.

Pause 2: The exploration done in the mathematizing sandbox leads students to the "a-ha moment" when they can match what they see happening in the problem to a problem situation (see Figures 1.2 and 1.3). Understanding the most appropriate problem situation informs which operation(s) to use, but it also does so much more. It builds a solid foundation of operation sense.

Step 3 (Express): Here students leave the sandbox and are ready to express the story either symbolically or in words or pictures, having found a solution they are prepared to discuss and justify.

LEVERAGING THE POWER OF CHILDREN'S LITERATURE

We just shared with you a problem-solving model for children to understand the context, or brief "story," as it is presented in a word problem. The model also offers guidance to work through what the problem means and how the numbers in it relate. Because children are naturally drawn to stories in order to make sense of their world, there are some additional, even broader ways to integrate the power of stories into helping students mathematize their world and build operation sense—using the kinds of literary stories found in children's books you already have in your classroom or school library. Throughout this book we will share examples, explaining how the following four strategies might work for you as you use children's literature to engage students in problem posing. You will come away with ideas for a few specific books, but, more important, you will consider how to use these strategies to explore books in your own school or classroom library to engage students in problem posing and problem solving.

STRATEGY #1: MAKE PREDICTIONS Many book-length stories rely on repetitive patterns or suspense in the story arc, and students learn to anticipate or predict what might happen. The Make Predictions strategy for problem posing pauses the action in a story, represents what's happening using a mathematical equation or representation, and gives students the opportunity to consider different ways the story might unfold. This strategy supports students' growing awareness of the structure of a story, using clues in the text to decide appropriate pivots in the story for predicting the outcomes.

STRATEGY #2: CREATE ANOTHER OUTCOME The events in some longer or more complex stories encourage readers to imagine a different way that the story might have unfolded. Maybe the main character makes a single decision that changes the whole story. Maybe an unfortunate event happens that might have been avoided. Or perhaps so much happens in the story that changing the story can be a great puzzle. Start with an initial reading, then mathematize the action in the story (identify quantities), and finally experiment with different changes to the story. This strategy supports literature standards that ask students to describe how characters in a story respond to major events and challenges.

STRATEGY #3: FIND THE UNKNOWN QUANTITY Some stories are told using quantities that are important to the story but are never given to the reader. In these stories, students have the opportunity to imagine quantities that make sense and write their own mathematical interpretations. Fairy tales, tall tales, and fables often rely on narratives of this kind, but rarely are actual numbers used. This strategy loans numbers to the story, or it invites students to consider different outcomes for the story using different quantities. This strategy, perhaps more than the others, relies on students to mathematize the details of a story. The strategy encourages them to use illustrations or other details in the story to gather information about the magnitude of the measurable or countable details of the story.

STRATEGY #4: TRANSCRIBE THE ACTION OR RELATIONSHIP Some stories give explicit quantities and the reader follows along as a quantity increases or decreases because of the action in the story (Monroe & Young, 2018). Many nursery rhymes, like *Ten Little Monkeys*, for example, rely on this narrative strategy. The mathematics lesson is designed to translate those changes into mathematical language. The students have the job of recording what happens to the quantities in the story using manipulatives, pictures, or equations in a way that is appropriate for their grade level. This strategy focuses student attention on the rhythm or structure of a story, using the details to quantify it or to translate it into a mathematical representation.

Final Words Before You Dive In

We understand that your real life in a school and in your classroom puts innumerable demands on your time and energy as you work to address ambitious mathematics standards. Who has time to use manipulatives, draw pictures, and spend time writing about mathematics? Your students do! This is what meeting the new ambitious standards actually requires. It may feel like pressure to speed up and do more, but paradoxically, the way to build the knowledge and concepts that are currently described in the standards is by slowing down. Evidence gathered over the past 30 years indicates that an integrated and connected understanding of a wide variety of representations of mathematical ideas is one of the best tools in a student's toolbox (or sandbox!) for a deep and lasting understanding of mathematics (Leinwand, Brahier, & Huinker, 2014). We hope that this book will be a valuable tool as you make or renew your commitment to teaching for greater understanding.